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The New Insight into the Theory of 2-D Complex and Quaternion Analytic Signals

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***Abstract*—The paper presents the overview of the theory of 2-D complex and quaternion analytic signals with the 1st-quadrant spectrum support. Both signals are expressed as complex/hypercomplex sums of partial and total 2-D Hilbert transforms. Moreover, starting with the definition of 2-D complex and quaternion Fourier transforms, the 2-D Hilbert transforms are derived in the form of sums of parts of different parity with respect to signal-domain variables. Some new relations for Hilbert quaternion spectra have been derived. The paper is illustrated with the example of the 2-D separable Cauchy signal.**

***Keywords*—2-D Fourier transform, Quaternion Fourier transform, 2-D Hilbert transforms, 2-D analytic signals.**

I. INTRODUCTION

THE 2-D complex analytic signals have been defined by Hahn [1] in the form of the 2-D inverse Fourier transform of the single quadrant spectrum of a 2-D real signal. Since the 2-D frequency space is divided into four quadrants, it is possible to define four different analytic signals. They are two and two conjugates which means that any 2-D real signal is in fact represented by two complex analytic signals with single quadrant spectra. The complete theory of multidimensional analytic signals has been presented in [2]. An alternative approach to the theory of analytic signals has been described in [3]. The authors of [3] defined the quaternion analytic signal as the inverse Quaternion Fourier transform of the single-quadrant quaternion spectrum of a 2-D real signal. However, it has been proved in [4] that both approaches (complex and quaternion) are equivalent and their choice is a matter of convenience.

The quaternion signals (not necessary analytic) have found numerous applications in digital signal processing, especially in color image processing. Some examples presented in Appendix A show their potential in different aspects. The 2-D analytic signals (complex and quaternion), as equivalent representations of a 2-D real signal, can be used to derive its polar form comprising the local amplitude/amplitudes and two or three phase functions. The respective formulas can be found in [4]. This problem is out of the scope of the paper. However, the information included in polar components of the signal is wealth to be studied to get better knowledge about the properties of a 2-D real signal. The possible area of practical applications is medical image processing where problems of orientation or edge detection are in spectrum of interest.

This paper focuses on some aspects of the theory of 2-D analytic (complex and quaternion) signals and shows the above mentioned equivalence in terms of even-odd components of 2-D total and partial Hilbert transforms defined by Hahn in [1]. The formulas relating complex and quaternion spectra of Hilbert transforms are derived.

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II. 2-D COMPLEX AND QUATERNION FOURIER TRANSFORM

A 2-D real signal can be written as a sum of even-even (*ee*), even-odd (*eo*), odd-even (*oe*) and odd-odd (*oo*) parts [2]:

$$u(x_2,x_1) = u_{ee}(x_2,x_1) + u_{eo}(x_2,x_1) + u_{oe}(x_2,x_1) + u_{oo}(x_2,x_1), \tag{1}$$

where

$$u_{ee}(\cdot) = \frac{u(x_2,x_1) + u(x_2,-x_1) + u(-x_2,x_1) + u(-x_2,-x_1)}{4}, \tag{2}$$

$$u_{eo}(\cdot) = \frac{u(x_2,x_1) - u(x_2,-x_1) + u(-x_2,x_1) - u(-x_2,-x_1)}{4}, \tag{3}$$

$$u_{oe}(\cdot) = \frac{u(x_2,x_1) + u(x_2,-x_1) - u(-x_2,x_1) - u(-x_2,-x_1)}{4}, \tag{4}$$

$$u_{oo}(\cdot) = \frac{u(x_2,x_1) - u(x_2,-x_1) - u(-x_2,x_1) + u(-x_2,-x_1)}{4}, \tag{5}$$

Notice that in (1)-(5) we applied the reversed order of subscripts: (x_2, x_1) instead of (x_1, x_2). It is due to the notation introduced in [1], [2], where e represented a binary 0 and o – a binary 1. It means that $u_{eo}(x_2, x_1)$ is an even function with respect to x_2 and an odd function with respect to x_1. The signs of terms in nominators in (2)-(5) are equal to products of odd-indexed variables. For example in (3), we have $-u(x_2, -x_1)$ and $-u(-x_2, -x_1)$ because this is an odd function with respect to x_1. On the other hand, in (5) we have $+u(-x_2, -x_1)$.

Fig. 1a shows the 2-D Cauchy signal: $u(x_2, x_1) = ab\Big/\Big(a^2 + (x_1 - c)^2\Big)\Big(b^2 + (x_2 - d)^2\Big)$ with $a = 1$, $b = 2$, $c = 1$, $d = 0.5$. Figures 1b-e illustrate respectively its even-even, even-odd, odd-even and odd-odd parts obtained directly from (2)-(5).

A. The 2-D Complex Fourier Transform of a 2-D Real Signal

The 2-D Fourier transform (2-D FT) of (1) has the form:

$$F\{u(x_2,x_1)\} = U(f_2,f_1) = \\ = \iint_{R^2} u(x_2,x_1) e^{-e_1\alpha_1} e^{-e_1\alpha_2} dx_2 dx_1 \tag{6}$$

where $\alpha_1 = 2\pi f_1 x_1$, $\alpha_2 = 2\pi f_2 x_2$ and the imaginary unit in the exponent usually denoted with i or j is here replaced

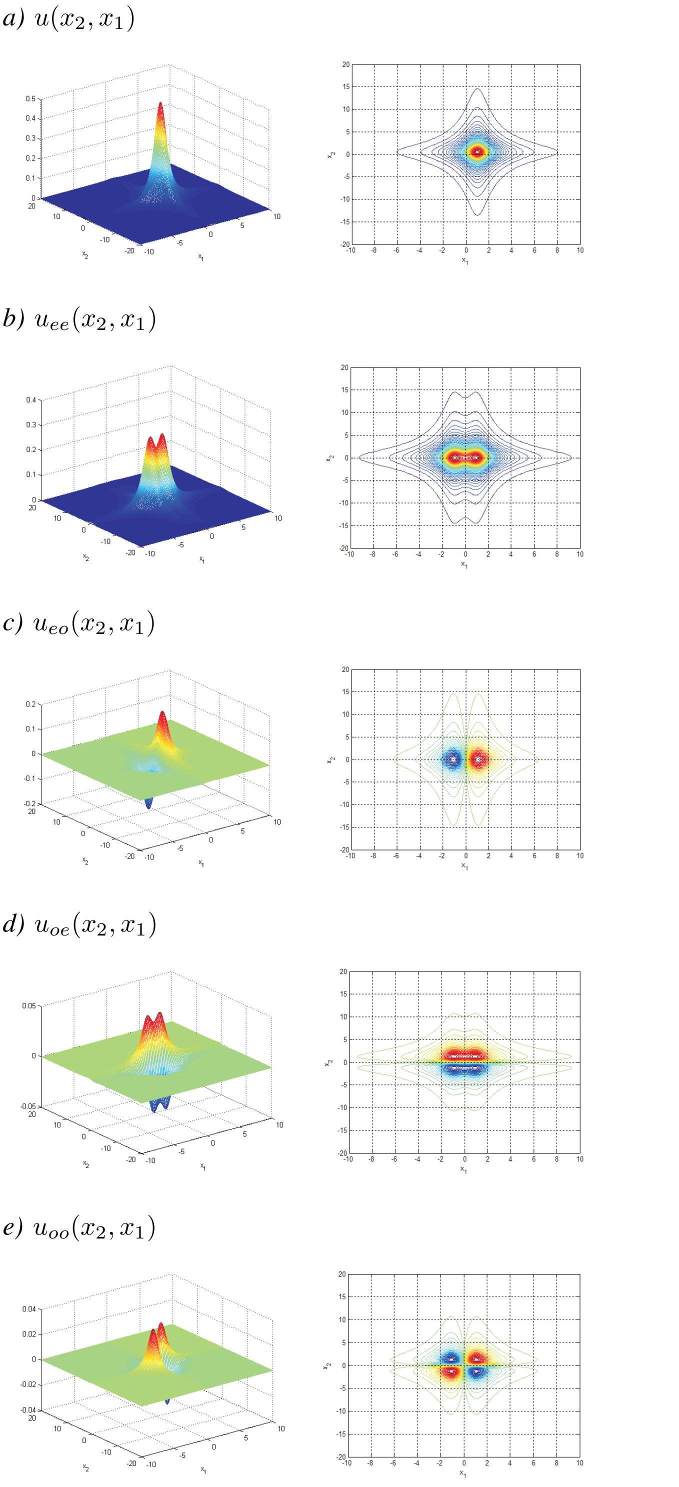

Fig. 1. a)-e): Mesh (left) and contour (right) plots of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

with e_1. As an example, the 2-D spectrum of the (separable) Cauchy signal from Fig. 1 is given by the complex function:

$$U(f_2, f_1) = \pi^2 \exp(-2\pi(a|f_1|+b|f_2|)) \exp(-j2\pi(cf_1+df_2)).$$

Fig. 2 shows its absolute value (amplitude spectrum).
The 2-D inverse FT is

$$F^{-1}\{U(f_2, f_1)\} = u(x_2, x_1) =$$

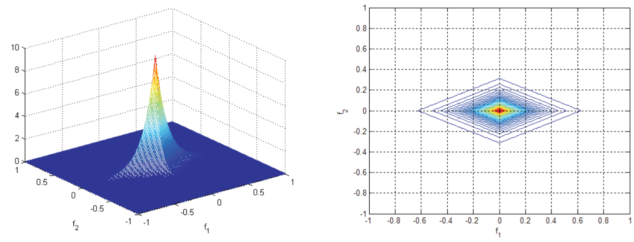

Fig. 2. Mesh (left) and contour (right) plots of the amplitude spectrum $|U(f_2, f_1)|$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

$$= \iint_{R^2} u(f_2, f_1) e^{e_1 \alpha_1} e^{e_1 \alpha_2} df_2 df_1. \tag{7}$$

The insertion of (1) into (6) yields the spectrum in the form of a complex sum of four terms:

$$U(f_2, f_1) = U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe}) \tag{8}$$

where

$$U_{ee}(\cdot) = \iint_{R^2} u_{ee}(x_2, x_1) \cos\alpha_2 \cos\alpha_1 dx_2 dx_1, \tag{9}$$

$$U_{eo}(\cdot) = \iint_{R^2} u_{eo}(x_2, x_1) \cos\alpha_2 \sin\alpha_1 dx_2 dx_1, \tag{10}$$

$$U_{oe}(\cdot) = \iint_{R^2} u_{oe}(x_2, x_1) \sin\alpha_2 \cos\alpha_1 dx_2 dx_1, \tag{11}$$

$$U_{oo}(\cdot) = \iint_{R^2} u_{oo}(x_2, x_1) \sin\alpha_2 \sin\alpha_1 dx_2 dx_1. \tag{12}$$

The 2-D spectrum (8) can be expressed in the equivalent form:

$$U(f_2, f_1) = \text{Re}(f_2, f_1) + e_1 \text{Im}(f_2, f_1) \tag{13}$$

where

$$\text{Re}(f_2, f_1) = U_{ee}(f_2, f_1) - U_{oo}(f_2, f_1), \tag{14}$$

$$\text{Im}(f_2, f_1) = -U_{eo}(f_2, f_1) - U_{oe}(f_2, f_1). \tag{15}$$

B. The Quaternion Fourier Transform of a 2-D Real Signal

There are various definitions of the Quaternion Fourier transform (QFT). We apply the definition introduced by Ell in [5] and called the two-sided QFT. The QFT and its inverse are given by

$$QFT\{u(x_2, x_1)\} = U_q(f_2, f_1) =$$

$$= \iint_{R^2} e^{-e_1 \alpha_1} u(x_2, x_1) e^{-e_2 \alpha_2} dx_2 dx_1, \tag{16}$$

$$QFT^{-1}\{U_q(f_2, f_1)\} = u(x_2, x_1) =$$

$$= \iint_{R^2} e^{e_1 \alpha_1} U_q(f_2, f_1) e^{e_2 \alpha_2} df_2 df_1. \tag{17}$$

The imaginary units e_1 and e_2 are elements of the basis $\{e_1, e_2, e_3\}$ of the algebra of quaternions $\mathbb{H}$ – the algebra of order 4 over the real numbers field $\mathbb{R}$. The basic properties of $\mathbb{H}$ are presented in Appendix A. The insertion of (1) into (16) yields

$$U_q(f_2, f_1) = U_{ee} - e_1 U_{eo} - e_2 U_{oe} + e_3 U_{oo} \tag{18}$$

where all terms are given by (9)-(12) and $e_3 = e_1 e_2$ according to the quaternion multiplication rules (see Appendix A).

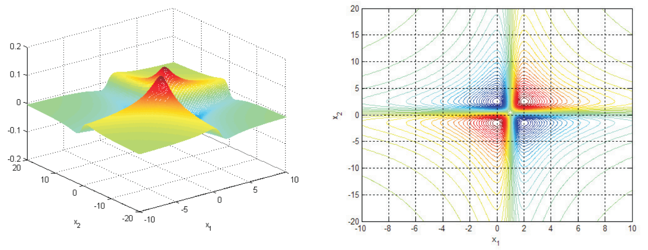

Fig. 3. Mesh (left) and contour (right) plots of $v(x_2, x_1)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

C. The Pei Formula

There exists the formula relating the two-sided QFT (16) and the 2-D FT (6) derived by Pei *et al.* in [6] and given by

$$U_q(f_2, f_1) = U(f_2, f_1)\frac{1 - e_3}{2} + U(-f_2, f_1)\frac{1 + e_3}{2}. \quad (19)$$

If we insert (13) into the above formula we get

$$U_q(f_2, f_1) = \mathrm{Re}(f_2, f_1)\frac{1 - e_3}{2} + \mathrm{Re}(-f_2, f_1)\frac{1 + e_3}{2} +$$

$$+ e_1\mathrm{Im}(f_2, f_1)\frac{1 - e_3}{2} + e_1\mathrm{Im}(-f_2, f_1)\frac{1 + e_3}{2} \quad (20)$$

It is easy to show that the insertion of (14) and (15) into (20) gives immediately the definition (18).

III. The 2-D Hilbert Transforms and Their Complex Spectra

In this section, we briefly recall the basic elements of the theory of 2-D analytic signals with single-quadrant spectra presented by Hahn in [1] and [2]. We focus on the 2-D complex analytic signal $\Psi_1(x_2, x_1)$ with a spectrum in the first quadrant of the frequency space defined in [1] as the 2-D inverse Fourier transform (7):

$$\Psi_1(x_2, x_1) = F^{-1}\{(1 + \mathrm{sgn} f_1)(1 + \mathrm{sgn} f_2)U(f_2, f_1)\} \quad (21)$$

In (21), the operator $\mathbf{1}(f_2, f_1) = (1 + \mathrm{sgn} f_1)(1 + \mathrm{sgn} f_2)$ is a 2-D unit step used to limit the spectrum to the 1^{st} quadrant of the frequency space. In [1] and [2], the definitions of other 2-D complex signals with spectra in next three quadrants are presented.

In the signal domain (x_2, x_1), the definition (21) corresponds to the double convolution:

$$\Psi_1(x_2, x_1) =$$

$$= u(x_2, x_1) * *(\delta(x_1) + e_1/\pi x_1)(\delta(x_2) + e_1/\pi x_2) \quad (22)$$

that is equivalent to

$$\Psi_1(x_2, x_1) = u - v + e_1(v_1 + v_2) \quad (23)$$

where

$$v(x_2, x_1) = F^{-1}\{-\mathrm{sgn} f_1 \mathrm{sgn} f_2 \cdot U(f_2, f_1)\} \quad (24)$$

is the 2-D *total Hilbert transform* w.r.t. (x_2, x_1) and

$$v_1(x_2, x_1) = F^{-1}\{-e_1 \mathrm{sgn} f_1 \cdot U(f_2, f_1)\}, \quad (25)$$

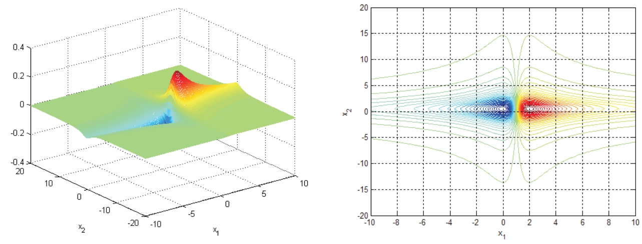

Fig. 4. Mesh (left) and contour (right) plots of $v_1(x_2, x_1)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

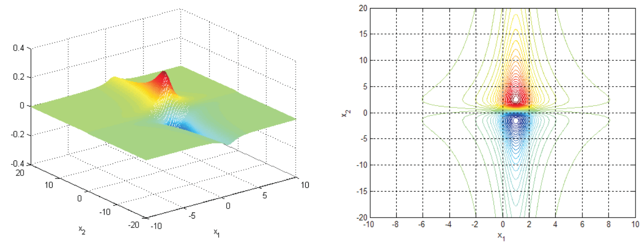

Fig. 5. Mesh (left) and contour (right) plots of $v_2(x_2, x_1)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

$$v_2(x_2, x_1) = F^{-1}\{-e_1 \mathrm{sgn} f_2 \cdot U(f_2, f_1)\} \quad (26)$$

are 2-D *partial Hilbert transforms* w.r.t. variables x_1 or x_2 respectively. In Figures 3-5, the total and partial Hilbert transforms of the 2-D Cauchy signal are presented. Due to the separability of the signal and the invariance of its Hilbert transforms in terms of translation in the signal domain, they are given by the formulas:

$$v(x_2, x_1) = (x_1 - c)(x_2 - d)\Big/\left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_1 - d)^2\right),$$

$$v_1(x_2, x_1) = b(x_1 - c)\Big/\left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_1 - d)^2\right),$$

$$v_2(x_2, x_1) = a(x_2 - d)\Big/\left(a^2 + (x_1 - c)^2\right)\left(b^2 + (x_1 - d)^2\right).$$

A. Spectra of Total and Partial Hilbert Transforms

Directly from (24)-(26) we get the 2-D Fourier spectra of total and partial Hilbert transforms:

$$V(f_2, f_1) = -\mathrm{sgn} f_1 \mathrm{sgn} f_2 \cdot U(f_2, f_1), \quad (27)$$

$$V_1(f_2, f_1) = -e_1 \mathrm{sgn} f_1 \cdot U(f_2, f_1), \quad (28)$$

$$V_2(f_2, f_1) = -e_1 \mathrm{sgn} f_2 \cdot U(f_2, f_1). \quad (29)$$

In Section III, the above formulas will be introduced into the Pei relation (19) to get the quaternion spectra of total and partial Hilbert transforms.

B. Even and Odd Terms of Total and Partial Hilbert Transforms (24)-(26)

Now, let us derive total and partial Hilbert transforms as sums of their even-even, even-odd, odd-even and odd-odd parts. We insert the Fourier transform $U(f_2, f_1)$ given by (8)

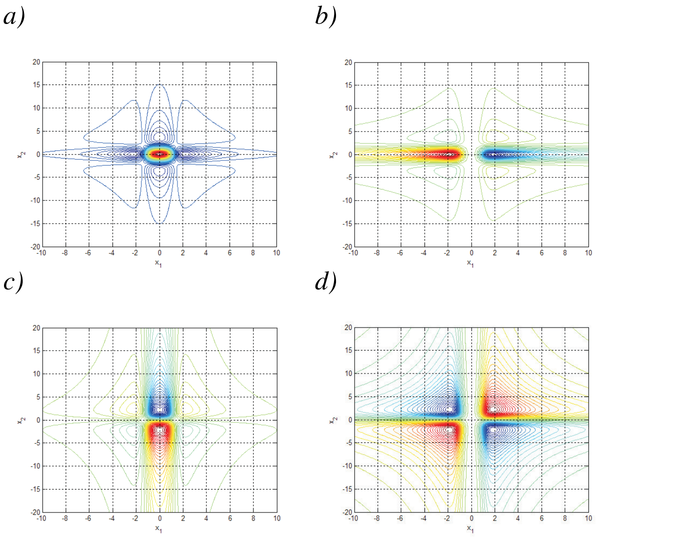

Fig. 6. Contour plots of (a) $v_{ee}(x_2, x_1)$, (b) $v_{eo}(x_2, x_1)$, (c) $v_{oe}(x_2, x_1)$, (d) $v_{oo}(x_2, x_1)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

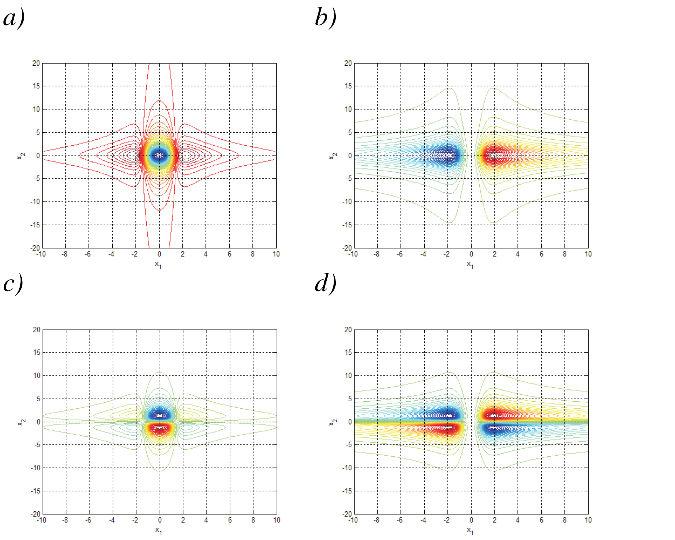

Fig. 7. Contour plots of (a) $v_{1ee}(x_2, x_1)$, (b) $v_{1eo}(x_2, x_1)$, (c) $v_{1oe}(x_2, x_1)$, (d) $v_{1oo}(x_2, x_1)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

into (24)-(26) and yield the following forms of the Hilbert transforms (the full derivation is presented in Appendix B):

$$v_1(x_2, x_1) = -v_{1_{ee}}^{(eo)} + v_{1_{eo}}^{(ee)} - v_{1_{oe}}^{(oo)} + v_{1_{oo}}^{(oe)}, \qquad (30)$$

$$v_2(x_2, x_1) = -v_{2_{ee}}^{(oe)} - v_{2_{eo}}^{(oo)} + v_{2_{oe}}^{(ee)} + v_{2_{oo}}^{(eo)}, \qquad (31)$$

$$v(x_2, x_1) = v_{ee}^{(oo)} - v_{eo}^{(oe)} - v_{oe}^{(eo)} + v_{oo}^{(ee)}. \qquad (32)$$

Let us note that in (30)-(32) superscripts indicate the even/odd parity of the corresponding term of $U(f_2, f_1)$ (8), while the subscripts – the even/odd parity of the corresponding Hilbert transform. Fig. 6 shows respectively the even-even, even-odd, odd-even and odd-odd parts of the total Hilbert transform of the 2-D Cauchy signal. In Figures 7 and 8, the decomposition of its partial Hilbert transforms into parts of different parity is visualized.

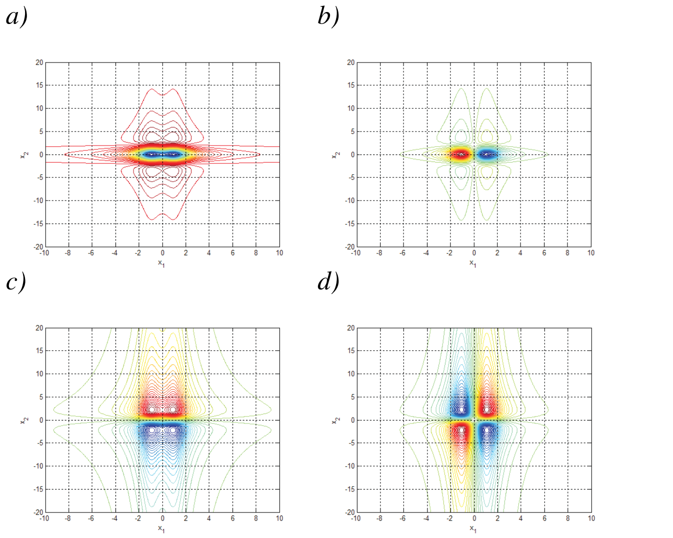

Fig. 8. Contour plots of (a) $v_{2ee}(x_2, x_1)$, (b) $v_{2eo}(x_2, x_1)$, (c) $v_{2oe}(x_2, x_1)$, (d) $v_{2oo}(x_2, x_1)$ of the 2-D Cauchy signal: $a = 1$, $b = 2$, $c = 1$, $d = 0.5$.

IV. The 2-D Hilbert Transforms and Their Quaternion Spectra

Again, let us recall the elements of the theory of the quaternion analytic signal defined in [3] as the 2-D inverse QFT (17) of the 1^{st} quadrant quaternion spectrum, i.e.:

$$\Psi_q(x_1, x_2) = QFT^{-1}\{(1 + \mathrm{sgn} f_1)(1 + \mathrm{sgn} f_2)U_q(f_2, f_1)\}, \qquad (33)$$

i.e., using the same single-quadrant operator as in (21). Using the definition of Ell [4] of the inverse QFT we have

$$\Psi_q(x_2, x_1) =$$
$$= \iint_{R^2} e^{e_1\alpha_1}(1 + \mathrm{sgn} f_1)(1 + \mathrm{sgn} f_2)U_q(f_2, f_1)e^{e_2\alpha_2} df_2 df_1. \qquad (34)$$

Let us remark that there is an alternative definition of the inverse QFT (called Right-side QFT) introduced by Ell [5] and also used by Hitzer [4]. It differs from (34) by the order of terms under the integral. In this case we have

$$\Psi_q(x_2, x_1) =$$
$$= \iint_{R^2} (1 + \mathrm{sgn} f_1)(1 + \mathrm{sgn} f_2)U_q(f_2, f_1)e^{e_2\alpha_2}e^{e_1\alpha_1} df_2 df_1. \qquad (35)$$

In our investigations, we use the definition (34) that corresponds in the signal domain to the convolution of the 2-D real signal $u(x_1, x_2)$ with the 2-D hypercomplex delta distribution [6]:

$$\Psi_q(x_2, x_1) =$$
$$= u(x_2, x_1) * *(\delta(x_1) + e_1/\pi x_1)(\delta(x_2) + e_2/\pi x_2) \qquad (36)$$

Exploding the above expression we get the definition of the quaternion analytic signal with the 1^{st} quadrant spectrum support:

$$\Psi_q(x_2, x_1) = u + e_1 v_1 + e_2 v_2 + e_3 v. \qquad (37)$$

We observe in (37) that the same Hilbert transforms as in (23) appear once again. In the next part, their quaternion spectra will be derived.

A. Quaternion Spectra of Total and Partial Hilbert Transforms

In order to derive the quaternion spectra of total and partial Hilbert transforms we apply the Pei formula (19). The quaternion spectra of v, v_1 and v_2 respectively are

$$V_q(f_2,f_1) = V(f_2,f_1)\frac{1-e_3}{2} + V(-f_2,f_1)\frac{1+e_3}{2}, \quad (38)$$

$$V_{1q}(f_2,f_1) = V_1(f_2,f_1)\frac{1-e_3}{2} + V_1(-f_2,f_1)\frac{1+e_3}{2}, \quad (39)$$

$$V_{2q}(f_2,f_1) = V_2(f_2,f_1)\frac{1-e_3}{2} + V_2(-f_2,f_1)\frac{1+e_3}{2} \quad (40)$$

where the Fourier spectra V, V_1 and V_2 are given by (27)-(29). Using the multiplication rules of the algebra of quaternions we get

$$V_q(f_2,f_1) =$$
$$= \mathrm{sgn}f_2\mathrm{sgn}f_1\Big(U_{ee}^{(oo)} + e_1U_{eo}^{(oe)} + e_2U_{oe}^{(eo)} + e_3U_{oo}^{(ee)}\Big), \quad (41)$$

$$V_{1q}(f_2,f_1) =$$
$$= \mathrm{sgn}f_1\Big(-U_{ee}^{(eo)} - e_1U_{eo}^{(ee)} + e_2U_{oe}^{(oo)} + e_3U_{oo}^{(oe)}\Big), \quad (42)$$

$$V_{2q}(f_2,f_1) =$$
$$= \mathrm{sgn}f_2\Big(-U_{ee}^{(oe)} + e_1U_{eo}^{(oo)} - e_2U_{ee}^{(eo)} + e_3U_{oo}^{(eo)}\Big). \quad (43)$$

In the above definitions the subscripts express the even/odd parity of the corresponding term and superscripts the even/odd parity of the term of the Fourier transform (compare with (9)-(12)). Such a double notation of indexes appeared to be very useful in derivations performed in Appendixes B and C.

It can be easily shown that using the definition (34) of the inverse QFT for spectra defined in (38)-(40) we get exactly the same forms of total and partial Hilbert transforms as in (30)-(32). The full derivation will be given in Appendix C.

Concluding we can state that complex and quaternion analytic signals are equivalent representations of the 2-D real signal $u(x_2,x_1)$ and only the mathematical apparatus used in derivations is different.

Moreover, it is possible to derive the formulas similar to (27)-(29) in the quaternion domain. They have the form

$$V_q(f_2,f_1) = (e_1\mathrm{sgn}f_1)U_q(f_2,f_1)(e_2\mathrm{sgn}f_2), \quad (44)$$

$$V_{1q}(f_2,f_1) = (-e_1\mathrm{sgn}f_1)U_q(f_2,f_1), \quad (45)$$

$$V_{2q}(f_2,f_1) = U_q(f_2,f_1)(-e_2\mathrm{sgn}f_2). \quad (46)$$

Note that the order of multiplication in the above formulas is strictly determined and cannot be changed due to the non-commutativity of the algebra of quaternions.

V. Properties of Hilbert Quaternion Spectra

It has been already shown in (37) that the quaternion analytic signal with the 1^{st}-quadrant spectrum support is $\Psi_q(x_2,x_1) = u + e_1v_1 + e_2v_2 + e_3v$. Applying the linearity of QFT for (37), we obtain

$$QFT\{\Psi_q(x_2,x_1)\} = QFT\{u + e_1v_1 + e_2v_2 + e_3v\} =$$
$$= QFT\{u\} + QFT\{e_1v_1\} + QFT\{e_2v_2\} + QFT\{e_3v\}. \quad (47)$$

However, from (33) we know that (47) should be equal to

$$QFT\{\Psi_q(x_2,x_1)\} = (1+\mathrm{sgn}f_1)(1+\mathrm{sgn}f_2)U_q(f_2,f_1) =$$
$$= U_q(\cdot) + \mathrm{sgn}f_1\cdot U_q(\cdot) + \mathrm{sgn}f_2\cdot U_q(\cdot) + \mathrm{sgn}f_1\mathrm{sgn}f_2\cdot U_q(\cdot) \quad (48)$$

Moreover, from (44)-(46) we get

$$e_1V_q(f_2,f_1)e_2 = \mathrm{sgn}f_1\mathrm{sgn}f_2\cdot U_q(f_2,f_1), \quad (49)$$

$$e_1V_{1q}(f_2,f_1) = \mathrm{sgn}f_1\cdot U_q(f_2,f_1), \quad (50)$$

$$V_{2q}(f_2,f_1)e_2 = \mathrm{sgn}f_2\cdot U_q(f_2,f_1). \quad (51)$$

Comparing (47) with (48) using relations (49)-(51) we get new relations for Hilbert quaternion spectra as follows:

$$QFT\{e_1v_1\} = e_1QFT\{v_1\} = e_1V_{1q}, \quad (52)$$

$$QFT\{e_2v_2\} = QFT\{v_2\}e_2 = V_{2q}e_2, \quad (53)$$

$$QFT\{e_3v\} = e_1QFT\{v\}e_2 = e_1V_qe_2. \quad (54)$$

To our knowledge, the relations (52)-(54) presented above seem to be an original result.

VI. Summary

In this paper, the properties of 2-D Hilbert transforms originally defined by Hahn in [1] have been studied in complex and quaternion domains. It has been proved, that both approaches give exactly the same result. Some new formulas relating complex and quaternion spectra have been developed and verified.

Appendix A
Algebra of Quaternions $\mathbb{H}$

The quaternions form a non-commutative algebra of order 4 over $\mathbb{R}$, denoted with $\mathbb{H}$. Basing on the Cayley-Dickson construction, a quaternion q is defined as an ordered pair of complex numbers z_0 and z_1 : $q = (z_0, z_1)$, where $z_0 = r_0 + r_1e_1$ and $z_1 = r_2 + r_3e_1$. Here again, we use e_1 for the imaginary unit (usually denoted with i and j). We can write

$$q = z_0 + z_1e_2 = (r_0 + r_1e_1) + (r_2 + r_3e_1)e_2. \quad (55)$$

Applying the Hamilton's multiplication rules of imaginary units in $\mathbb{H}$ (Table I), we obtain the general form of a quaternion:

$$q = r_0 + r_1\cdot e_1 + r_2\cdot e_2 + r_3\cdot e_3. \quad (56)$$

The conjugate of q and its norm $|q|$ respectively are

$$q^* = r_0 - r_1\cdot e_1 - r_2\cdot e_2 - r_3\cdot e_3. \quad (57)$$

TABLE I
MULTIPLICATION OF IMAGINARY UNITS IN Ç

$\times$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	-e_2
e_2	e_2	-e_3	-1	e_1
e_3	e_3	e_2	-e_1	-1

$$|q| = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}. \qquad (58)$$

The quaternions (56) with $r_0 = 0$ are referred as *pure (reduced)* quaternions and those with $|q| = 1$ as *unit (unitary)* quaternions.

Hypercomplex numbers have found numerous applications in different fields, e.g., in color image processing [7]–[11] and in digital signal processing [12]–[15] In color image processing, reduced quaternions and reduced commutative biquaternions [16] are used as representations of a RGB image. Its spectrum is calculated using the Quaternion FT introduced and used by Ell [5], [9] or its discrete version called Quaternion Discrete FT (QDFT) [7] which opens up a very wide range of possible applications [6], [10], [11]. In [10] for example, the problem of estimation of the motion characteristics within a time-varying color image scene is investigated. It has been shown that using hypercomplex (quaternion) approach results in lower computational cost and better performance in presence of different distortions. Another problem is the edge detection in color images using filters based on convolution with quaternion masks or the Discrete Quaternion Correlation function [6], [11]. The hypercomplex representation of a color image makes easy detecting objects with the same shape, size, color and brightness as the reference pattern (or presenting only two common features, like shape and color or shape and brightness).

In digital signal processing quaternions and reduced biquaternions are applied in filter design [13], [14]. Using the hypercomplex approach, it is possible to reduce the order of a real filter and to process vector-valued signals (functions of a few independent parameters) as a whole conveying information (e.g., directivity) which is lost in case of treating each component separately.

APPENDIX B
HILBERT TRANSFORMS DERIVED USING A 2-D COMPLEX SIGNAL WITH THE 1^{ST} QUADRANT SPECTRUM SUPPORT

In this part, we derive the Hilbert transforms (total and partial) starting with the decomposition of the spectrum of the real signal $u(x_2, x_1)$ into its even-even, even-odd, odd-even and odd-odd parts (7). Substituting the Fourier spectrum (7) into (22), we obtain

$$v(x_2, x_1) =$$
$$= -\iint_{R^2} \mathrm{sgn}(f_2)\mathrm{sgn}(f_1)$$
$$[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})]e^{e_1(\alpha_1+\alpha_2)}df =$$
$$= \iint_{R^2} \mathrm{sgn}(f_2)\mathrm{sgn}(f_1)U_{ee}s_2s_1df +$$
$$+ \iint_{R^2} \mathrm{sgn}(f_2)\mathrm{sgn}(f_1)U_{oo}c_2c_1df +$$
$$- \iint_{R^2} \mathrm{sgn}(f_2)\mathrm{sgn}(f_1)U_{eo}s_2c_1df +$$
$$- \iint_{R^2} \mathrm{sgn}(f_2)\mathrm{sgn}(f_1)U_{oe}c_2s_1df \qquad (59)$$

where $c_1 = \cos(2\pi f_1)$, $c_2 = \cos(2\pi f_2)$, $s_1 = \sin(2\pi f_1)$, $s_2 = \sin(2\pi f_2)$ and $f = (f_2, f_1)$. After reordering the terms we finally get

$$v(x_2, x_1) = v_{ee}^{(oo)} - v_{eo}^{(oe)} - v_{oe}^{(eo)} + v_{oo}^{(ee)}. \qquad (60)$$

To get the partial Hilbert transform v_1 expressed as the sum of terms of different parity, we substitute the spectrum given by (8) into (22):

$$v_1(x_2, x_1) =$$
$$= -e_1 \iint_{R^2} \mathrm{sgn}(f_1)[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})]e^{e_1(\alpha_1+\alpha_2)}df =$$
$$= \iint_{R^2} \mathrm{sgn}(f_1)U_{ee}c_2s_1df - \iint_{R^2} \mathrm{sgn}(f_1)U_{oo}s_2c_1df +$$
$$- \iint_{R^2} \mathrm{sgn}(f_1)U_{eo}c_2c_1df + \iint_{R^2} \mathrm{sgn}(f_1)U_{oe}s_2s_1df \quad (61)$$

The partial Hilbert transform v_1 is equal to

$$v_1(x_2, x_1) = -v_{1_{ee}}^{(eo)} + v_{1_{eo}}^{(ee)} - v_{1_{oe}}^{(oo)} + v_{1_{oo}}^{(oe)}. \qquad (62)$$

Analogously, repeating the same procedure as above we get the partial Hilbert transform v_2. From (9) and (22) we obtain

$$v_2(x_2, x_1) =$$
$$= -e_1 \iint_{R^2} \mathrm{sgn}(f_2)[U_{ee} - U_{oo} - e_1(U_{eo} + U_{oe})]e^{e_1(\alpha_1+\alpha_2)}df =$$
$$= \iint_{R^2} \mathrm{sgn}(f_2)U_{ee}s_2c_1df - \iint_{R^2} \mathrm{sgn}(f_2)U_{oo}c_2s_1df +$$
$$- \iint_{R^2} \mathrm{sgn}(f_2)U_{eo}c_2c_1df - \iint_{R^2} \mathrm{sgn}(f_2)U_{oe}s_2s_1df \quad (63)$$

and finally

$$v_2(x_2, x_1) = -v_{2_{ee}}^{(oe)} - v_{2_{eo}}^{(oo)} + v_{2_{oe}}^{(ee)} + v_{2_{oo}}^{(eo)}. \qquad (64)$$

APPENDIX C
HILBERT TRANSFORMS DERIVED USING A QUATERNION SIGNAL WITH THE 1^{ST} QUADRANT SPECTRUM SUPPORT

Let us derive the total and partial Hilbert transforms in a different way as in Appendix B. We know that they are successive imaginary terms of the quaternion analytic signal given by (35). Inserting into (32) the quaternion spectrum U_q given by (16) we get

$$\Psi_q(x_2, x_1) = \iint_{R^2} e^{e_1\alpha_1}\mathbf{1}(f_2, f_1)U_q(f_2, f_1)e^{e_2\alpha_2}df =$$
$$= \iint_{R^2} e^{e_1\alpha_1}\mathbf{1}(f_2, f_1)(U_{ee} - e_1U_{eo} - e_2U_{oe} + e_3U_{oo})e^{e_2\alpha_2}df =$$
$$= \iint_{R^2} (c_1 + e_1s_1)\mathbf{1}(f_2, f_1)$$

$$(U_{ee} - e_1 U_{eo} - e_2 U_{oe} + e_3 U_{oo})(c_2 + e_2 s_2) df \quad (65)$$

where $\mathbf{1}(f_2, f_1) = (1 + \text{sgn} f_1)(1 + \text{sgn} f_2)$ and the notations are the same as introduced in (59). The real part of (65) is

$$u(x_2, x_1) = \text{Re}\{\Psi_q(x_2, x_1)\} =$$
$$= \iint_{R^2} (U_{ee} c_2 c_1 + U_{eo} c_2 s_1 + U_{oe} s_2 c_1 + U_{oo} s_2 s_1) df =$$
$$= u_{ee}(x_2, x_1) + u_{eo}(x_2, x_1) + u_{oe}(x_2, x_1) + u_{oo}(x_2, x_1) \quad (66)$$

and coincides with (1). Next, the partial Hilbert transform v_1 is equal to

$$v_1(x_2, x_1) =$$
$$= \iint_{R^2} \text{sgn} f_1 (U_{ee} c_2 s_1 - U_{eo} c_2 c_1 + U_{oe} s_2 s_1 - U_{oo} s_2 c_1) df =$$
$$= -v_{1_{ee}}^{(eo)} + v_{1_{eo}}^{(ee)} - v_{1_{oe}}^{(oo)} + v_{1_{oo}}^{(oe)}. \quad (67)$$

The partial Hilbert transform v_2 is

$$v_2(x_2, x_1) =$$
$$= \iint_{R^2} \text{sgn} f_2 (U_{ee} s_2 c_1 + U_{eo} s_2 s_1 - U_{oe} c_2 c_1 - U_{oo} c_2 s_1) df =$$
$$= -v_{2_{ee}}^{(oe)} - v_{2_{eo}}^{(oo)} + v_{2_{oe}}^{(ee)} + v_{2_{oo}}^{(eo)}. \quad (68)$$

and finally the total Hilbert transform v is given by

$$v(x_2, x_1) =$$
$$= \iint_{R^2} \text{sgn} f_2 \text{sgn} f_1 \cdot$$
$$\cdot (U_{ee} s_2 s_1 - U_{eo} s_2 c_1 - U_{oe} c_2 s_1 + U_{oo} c_2 c_1) df =$$
$$= v_{ee}^{(oo)} - v_{eo}^{(oe)} - v_{oe}^{(eo)} + v_{2_{oo}}^{(ee)}. \quad (69)$$

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