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On Definition of Operator *o* for Weakly Nonlinear Circuits

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***Abstract*—For the first time, operator *o* appeared in the literature on weakly nonlinear circuits in a Narayanan's paper on modelling transistor nonlinear distortion with the use of Volterra series. Its definition was restricted only to the linear part of a nonlinear circuit description. Obviously, as we show here, Narayanan's operator *o* had meaning of a linear convolution integral. The extended version of this operator, which was applied to the whole nonlinear circuit representation by the Volterra series, was introduced by Meyer and Stephens in their paper on modelling nonlinear distortion in variable-capacitance diodes. We show here that its definition as well as another definition communicated to the author of this paper are faulty. We draw here attention to these facts because the faults made by Meyer and Stephens were afterwards replicated in publications of Palumbo and his coworkers on harmonic distortion calculation in integrated CMOS amplifiers, and recently in a paper about distortion analysis of parametric amplifier by H. Shrimali and S. Chatterjee. These faults are also present in some class notes for students, which are available on WWW-pages.**

***Keywords*—Operator *o*, descriptions of mildly nonlinear circuits in the frequency-domain, nonlinear distortion, Volterra series.**

I. Introduction

To our best knowledge, an operator denoted shortly as *o* appeared for the first time in the literature in [1] in the context of the nonlinear distortion analysis in bipolar transistor circuits with the use of Volterra series [2]. In [1], this operator was defined strictly as a linear operator, associated with the linear impedance. Moreover, it was assumed to be an operator working in the time domain. Furthermore, its definition was not extended by Narayanan to a nonlinear case in [1], albeit really a nonlinear problem of calculating nonlinear distortion was considered in [1]. Only a linear part of the models used for the circuit analysis was described by Narayanan with the use of operator *o*, but their strictly nonlinear part in another way.

Referring to as the derivations presented in [1], Meyer and Stephens interpreted incorrectly [3] the operator *o* as one that enables an input-output circuit description in a mixed way, that is with the use of voltages and currents in the time domain and functions describing a circuit in the multi-dimensional frequency domain. The expression for such the mixed way of description of a nonlinear circuit behaviour, which they have given in [3] referring to as [1], cannot be however found in [1]. Even worse, Meyer and Stephens by publishing their ambiguous expression caused that many researchers afterwards begin to believe that a general description of a nonlinear circuit in the aforementioned mixed form does exist. Among them, there were the authors of the papers [4-7], and now such the belief seems to be very common either in research papers or didactic materials for students, see, for example, [8] and [9], respectively. In [8], the operator *o* became even an ordinary multiplication.

The problem sketched above has been already discussed briefly by the author of this paper at oral presentation and in a conference paper [10]. However, not all of its aspects have been addressed. Therefore, they need in our opinion more explanation.

The remainder of this paper is organized as follows. In the next section, we show thoroughly that *o* in [1] denotes in fact only a convolution integral, nothing more. In section III, we consider an imprecise definition [3] of the operator *o* and show that a Volterra series applying this operator does not in fact exist. Next, the corrected representation for description of a mildly nonlinear circuit or system is presented. Section IV is devoted to a new interpretation by Meyer of his *o* operator definition, which was recently communicated to the author of this paper [14]. It is shown here that this new interpretation is also faulty. The paper ends with some conclusions.

II. Meaning of Operator *o* in Work of Narayanan

Narayanan in his first paper [1] in a series of articles on nonlinear distortion analysis in bipolar transistor circuits with the use of the Volterra series [2] introduced an operator (operation) *o*. Referring to an equivalent nonlinear circuit of the common-emitter bipolar transistor connection shown below in Fig. 1, he simply said that "the impedances are represented by their transforms and *o* denotes that it operates on the voltage across it" [1, page 1000 therein]. And nothing more about the operator *o*.

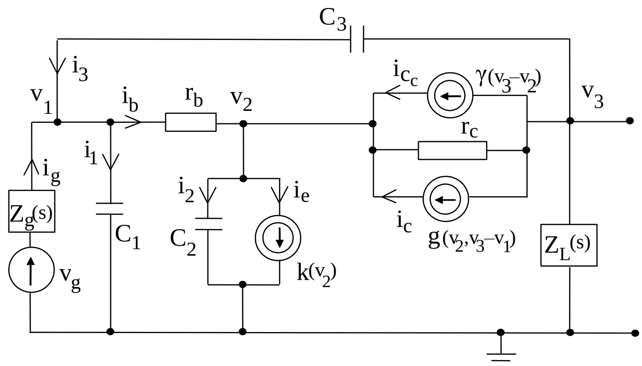

Fig. 1. Equivalent nonlinear circuit for the common-emitter bipolar transistor connection after Narayanan [1].

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Note that to be precise in our reference to [1] the original notation from Narayanan's article regarding the elements of the equivalent circuit in Fig. 1 was retained in this figure. But, for more details connected with the construction of the equivalent scheme shown in Fig. 1 and terminology used, the interested reader is referred to [1].

For proceeding further to deduce what the above Narayanan's descriptive definition of the operator o does really mean, let us yet rewrite here the nodal equations for the circuit of Fig. 1 formulated in [1, page 1000]. So, in this case, we have

$$\frac{1}{Z_g(s)} o\left(v_g - v_1\right) + \left(sC_3\right) o\left(v_3 - v_1\right) = \left(sC_1\right) o v_1 + \frac{1}{r_b} o\left(v_1 - v_2\right), \quad (1)$$

$$\frac{1}{r_b} o\left(v_1 - v_2\right) = k\left(v_2\right) + \left(sC_2\right) o v_2 + \frac{1}{r_c} o\left(v_2 - v_3\right) - \gamma\left(v_3 - v_2\right) - g\left(v_2, v_3 - v_1\right), \quad (2)$$

$$-\gamma\left(v_3 - v_2\right) + \frac{1}{r_c} o\left(v_2 - v_3\right) - g\left(v_2, v_3 - v_1\right) = \left(sC_3\right) o\left(v_3 - v_1\right) + \frac{1}{Z_L(s)} o v_3, \quad (3)$$

where $k\left(v_2\right)$, $\gamma\left(v_3 - v_2\right)$, and $g\left(v_2, v_3 - v_1\right)$ are the nonlinear current sources that depend upon the corresponding voltages.

Looking at the form of equations (1), (2), and (3), we observe that the operator o is used in them solely in connection with the linear elements occurring in the equivalent circuit scheme of Fig. 1. And in Fig. 1, we have the following linear circuit elements: resistors r_b and r_c; capacitors C_1, C_2, and C_3; input generator impedance $Z_g(s)$ and output load impedance $Z_L(s)$.

Further, we do not see in (1), (2), or (3) any application of the operator o to the nonlinear current sources $k\left(v_2\right)$, $\gamma\left(v_3 - v_2\right)$, and $g\left(v_2, v_3 - v_1\right)$. Hence, it follows that o is a strictly linear operator associated exclusively with the linear circuit elements. As such, it is, generally saying, a convolution integral operator of the form

$$y(t) = (NE) o x(t) = \int_{-\infty}^{\infty} h_{NE}(\tau) x(t - \tau) d\tau \quad (4)$$

where t denotes a time variable, $x(t)$ means the voltage across a given circuit element (or current flowing through it), and the meaning of $y(t)$ is just opposite. The symbol NE in (4) is used to denote the "name of a given circuit element". And finally, $h_{NE}(t)$ is the so-called impulse response (of a given element).

Comparing notation associated with the operator o used in (4) and by Narayanan, which is repeated in (1), (2), or (3), we see that the following element names: $1/r_b$ and $1/r_c$; sC_1, sC_2, and sC_3; $1/Z_g(s)$ and $1/Z_L(s)$ are associated with the resistors r_b and r_c; capacitors C_1, C_2, and C_3; input generator impedance $Z_g(s)$ and output load impedance $Z_L(s)$, respectively.

Observe now that a more convenient notation than that used by Narayanan in [1] could be applied by the use of a name associated, in some way, with the name of the function occurring on the right–hand side of (4), for example, as

$$y(t) = H_{NE}\left(x(t)\right) = \int_{-\infty}^{\infty} h_{NE}(\tau) x(t - \tau) d\tau . \quad (5)$$

The convention presented in (5) is used in the theory of systems and operators. Moreover, in the above context, observe also, that precisely saying, o stands only for carrying out an operation of integral convolution. One of the operands of this operation is the circuit element impulse response $h_{NE}(t)$, as expressed in (4) and (5). This is the cause that (4) and (5) can be also written equivalently in the following way

$$y(t) = \left(h_{NE} o x\right)(t) = \left(h_{NE} * x\right)(t) = \left(h_{NE} \otimes x\right)(t), \quad (6)$$

where the symbols "o", "$*$", and "$\otimes$" stand alternatively for the convolution operation. Note that the latter two are widely used in the research literature and textbooks on the linear theory and signal processing for denoting convolution integral operation.

It can be shown by exploiting only some basics of linear circuit theory that the impulse responses of the circuit resistive and capacitive elements occurring in Fig. 1 assume the following forms

$$h_{g_b}(t) = \frac{1}{r_b} \delta(t) \quad \text{and} \quad h_{g_c}(t) = \frac{1}{r_c} \delta(t), \quad (7)$$

and

$$h_{C_i}(t) = C_i^{-1} \times 1(t) = \frac{1}{C_i} \text{ for } t \geq 0 \text{ and } 0 \text{ for } t < 0, \quad i = 1, 2, 3, \quad (8)$$

for example, see [11]. In (7), $\delta(t)$ means the Dirac impulse. Moreover, $g_b = 1/r_b$ and $g_c = 1/r_c$ in (7) mean the corresponding conductances associated with the resistors r_b and r_c, respectively. Further, the symbol $1(t)$ in (8) stands for the Heaviside unit step function.

Regarding the impedances $Z_g(s)$ and $Z_L(s)$, or more conveniently, their admittances $Y_g(s) = 1/Z_g(s)$ and $Y_L(s) = 1/Z_L(s)$, respectively, (as they are in fact used in formulation of the nodal equations (1), (2), and (3)), we shall present, in what follows, some impulse responses for them for some concrete forms of the above impedances. Let us first consider an example of the impedance $Z_g(s) = r + sL$, with *r* meaning a resistance connected in series with an inductance *L*. Hence, the equivalent admittance will have in this case the following form

$$Y_g(s) = \frac{1}{Z_g(s)} = \frac{1}{r+sL} = \frac{1}{L}\frac{1}{s+\frac{r}{L}} \; . \tag{9}$$

For such the form of the transform as occurring on the right-hand side of (9), we get from a table of Laplace transforms (see, for example, [12]) the following function of time

$$h_{Y_g}(t) = \frac{1}{L} e^{-\frac{r}{L}t} \cdot 1(t) \; . \tag{10}$$

As a second illustrative example, consider now the impedance of a series connection of a resistance *r*, an inductance *L*, and a capacitance *C*. For this connection, we will have $Z_g(s) = r + sL + 1/(sC)$ or equivalently as the admittance

$$Y_g(s) = \frac{1}{Z_g(s)} = \frac{1}{L}\frac{s}{s^2 + s\frac{r}{L} + \frac{1}{LC}} \; . \tag{11}$$

Looking at the table [11] of the Laplace transforms, we get for (11) the inverse transform as

$$h_{Y_g}(t) = \frac{1}{L} e^{-\alpha t}\left(\cos(\omega t) - \frac{\alpha}{\omega}\sin(\omega t)\right)\cdot 1(t) \tag{12}$$

with the coefficients $\alpha = r/(2L)$ and $\omega = \sqrt{1/(LC) - (r/2L)^2}$.

So, concluding, we can say that depending upon the form of the impedance $Z_g(s)$, we easily find the associated operand $h_{Y_g}(t)$ - for performing the convolution operation related to it - by using the procedure sketched above. Moreover, the same regards the impedance $Z_L(s)$.

III. Meaning of Operator *O* in Work of Meyer and Stephens

In their paper [3], Meyer and Stephens claim that Narayanan in [1] has derived a special Volterra series representation, which, referring to an equivalent circuit of Fig. 1, would allow to describe the relation between the circuit output voltage $v_3(t)$ and its input current $i_g(t)$ as

$$\begin{aligned} v_3(t) &= A_1(f)\, o i_g(t) + A_2(f_1, f_2)\, o i_g^2(t) + \\ &+ A_3(f_1, f_2, f_3)\, o i_g^3(t) + \ldots \end{aligned} \; , \tag{13}$$

where $A_1(f)$, $A_2(f_1, f_2)$, and $A_3(f_1, f_2, f_3)$ mean the nonlinear (current-to-voltage) transfer functions of the circuit of Fig. 1 of the first, second, and third order, respectively. In [3], these transfer functions are called the Volterra coefficients. Obviously, they are the one-, two-, and three-dimensional Fourier transforms of the corresponding nonlinear circuit impulse responses of the first-, second-, and third-order [13], accordingly. Regarding the operator *o* used in (13), Meyer and Stephens say in [3, page 47] that "the operator sign indicates that the magnitude and phase of each term in i_g^n is to be changed by the magnitude and phase of $A_n(f_1, f_2, \ldots, f_n)$". As we know the operation of convolution does this, when we transform it to the frequency domain. But, it should be mentioned now (what was not done in [3]) that the symbol *o* in (13) has slightly different meanings in the consecutive components on the right-hand side of (13). Namely, it means subsequently the one-, two-, and three-dimensional convolution integrals, on the contrary to its definition in [1], where it meant only one-dimensional convolution integral. Furthermore, bad news for [3] is also that it is impossible at all to find a Volterra series description like that given by (13) in the Narayanan's work [1].

We shall show now that such a representation as given by (13) does not exist at all, even in the linear case. To this end, let us write

$$(a_1 o i_g)(t) = (a_1 \otimes i_g)(t) = \int_{-\infty}^{\infty} a_1(\tau) i_g(t-\tau)\, d\tau \tag{14}$$

with $a_1(t)$ meaning the first-order (liner) impulse response of the circuit in Fig. 1. Next, note that (14) would correspond to the first component on the right-hand side of (13).

Further, we introduce the Fourier inverse transform of $A_1(f)$ given by

$$a_1(t) = \int_{-\infty}^{\infty} A_1(f)\exp(j2\pi f t) df \tag{15}$$

into (14). This leads to

$$\left(a_1 o i_g\right)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1(f)\exp(j2\pi f\tau) df\, i_g(t-\tau)\, d\tau , \quad (16)$$

which after rearranging the terms and introducing a new variable $\tau' = t - \tau$ gives

$$\begin{aligned}\left(a_1 o i_g\right)(t) &= -\int_{-\infty}^{\infty} A_1(f)\exp(j2\pi ft)\cdot \\ &\cdot \int_{\infty}^{-\infty} i_g(\tau')\exp(-j2\pi f\tau')d\tau' df\end{aligned} . \quad (17)$$

Further, recognizing in (17) a Fourier transform for $i_g(t)$ (denote it by $I_g(f)$), we can rewrite (17) as

$$\left(a_1 o i_g\right)(t) = \int_{-\infty}^{\infty} A_1(f) I_g(f)\exp(j2\pi ft)df . \quad (18)$$

So, finally, we see that (18) represents nothing else than an inverse Fourier transform of $A_1(f)I_g(f)$.

Using a more compact notation $F^{-1}\{\cdot\}$ for the inverse Fourier transform, we can rewrite (18) as

$$\left(a_1 o i_g\right)(t) = F^{-1}\{A_1(f)I_g(f)\} \quad (19)$$

Comparing now the right-hand side expression in (19) with $A_1(f)oi_g(t)$ in (13), we see that they differ from each other. And the first makes sense, but the latter not.

Note now that the same argumentation can be applied to the second, third, and all the further components on the right-hand side of (13). So, this allows us to write a correct version of (13) in the following way

$$\begin{aligned}v_3(t) &= F_1^{-1}\{A_1(f)I_g(f)\} + \\ &+ F_2^{-1}\{A_2(f_1,f_2)I_g(f_1)I_g(f_2)\} + \\ &+ F_3^{-1}\{A_3(f_1,f_2,f_3)I_g(f_1)I_g(f_2)I_g(f_3)\} + \ldots\end{aligned} , \quad (20)$$

where $F_i^{-1}\{\cdot\}$, $i = 1,2,3,\ldots$, means the inverse i-dimensional Fourier transform.

Substituting $I_g(f) = F_{1f}(i_g(t))$ in (20), we get finally

$$\begin{aligned}v_3(t) &= F_1^{-1}\{A_1(f)F_{1f}(i_g(t))\} + \\ &+ F_2^{-1}\{A_2(f_1,f_2)F_{1f_1}(i_g(t))F_{1f_2}(i_g(t))\} + \\ &+ F_3^{-1}\Big\{A_3(f_1,f_2,f_3)F_{1f_1}(i_g(t))F_{1f_2}(i_g(t))\cdot \\ &\cdot F_{1f_3}(i_g(t))\Big\} + \ldots \quad ,\end{aligned} \quad (21)$$

where $F_{1f_z}\{\cdot\}$ stands for the one-dimensional Fourier transform, in which the frequency variable is denoted as f_z, $z = 1,2,3,\ldots$.

Observe now that the description (21) resembles the mixed time-frequency representation given by (13). In fact, the former is a correct version of the latter.

Moreover, observe also that (21) is nothing else than a Volterra series in the time domain, now with the Fourier transforms used in it.

IV. NEW EXPLANATION OF OPERATOR *O* BY MEYER

Prof. R. Meyer has been informed by the author of this paper about a problem with his definition of the operator *o*. He received an earlier version of this paper describing the problem in detail. His answer [14] was as follows (applying our notation used in the previous sections): „We introduced an operator *o* in our paper that can be defined precisely as follows. Let $A_n(j2\pi f_1, j2\pi f_2,\ldots, j2\pi f_n)$ be a function of complex arguments $j2\pi f_1, j2\pi f_2,\ldots, j2\pi f_n$. Let $K\sin\big(2\pi(f_{s1}+f_{s2}+\ldots+f_{sn})t+\varphi\big)$ be a sinusoid of amplitude *K*, frequency $(f_{s1}+f_{s2}+\ldots+f_{sn})$ and phase φ. Then

$$\begin{aligned}&A_n(j2\pi f_1, j2\pi f_2,\ldots, j2\pi f_n)o\Big[K\sin\big(2\pi(f_{s1}+f_{s2}+ \\ &+\ldots+f_{sn})t+\varphi\big)\Big] = \left|A(j2\pi f_{s1}, j2\pi f_{s2},\ldots j2\pi f_{sn})\right|\cdot \\ &\cdot K\sin\big(2\pi(f_{s1}+f_{s2}+\ldots+f_{sn})t+ \\ &+\varphi + arg A(j2\pi f_{s1}, j2\pi f_{s2},\ldots, j2\pi f_{sn})\big) \quad .\end{aligned} \quad (22)$$

This definition defines an unambiguous mapping from the field of arbitrary sine waves to another field of sine waves. If this definition is rigorously followed (as in our paper) the effects of weak nonlinearities in causing distortion in electronic circuits can be (and were) accurately calculated."

In what follows, we will show that also this definition is not correct for $n > 1$. First, however, let us make some remarks regarding the above refined definition of prof. Meyer.

Remark 1. The original definition that was published in [3] is directly related with the Volterra series and its theory. Simply, the Volterra series is formulated in [3] with the use of the operator *o*. The definition formulated in [14] is a refined and revised version of the former one. Therefore, the function

$A_n\left(j2\pi f_1, j2\pi f_2,\ldots, j2\pi f_n\right)$ of the complex arguments mentioned before cannot be considered in isolation from the Volterra theory. In this context, it means nothing else than the *n*-th order nonlinear transfer function of a nonlinear system.

Remark 2. Meyer's new definition of the operator *o* in (22) is also faulty. Simply, such and operator related with the Volterra series does not exist for $n>1$. We will show this in detail later. Obviously, one can arbitrary define such the operator *o* as in (22) (or some other one), but it will be useless in defining the Volterra series correctly.

Remark 3. The analysis carried out in [3] is correct. However, this is a result of consequently using the well-known form of the Volterra series as defined, for example, in [13]. Not by using a faulty operator *o*. Getting the results obtained in [3] would not be simply possible using this incorrectly defined operator *o*.

Remark 4. Two Meyer's definitions of the operator *o*: one formulated in [3] and next given in [14] are not identical. The former is more general because it was formulated for any signals, but the latter exclusively for a specific class of signals, sinusoidal ones. In other words, the range of validity of a model using the operator *o* defined in [3] would be wider than that of the model specified in [14] if these definitions were correct.

Let us consider now the case associated with putting $n=1$ (a linear one). It will be treated here separately because, as we will see later, it leads to some other results as those we get for $n>1$. Obviously, the case of $n=1$ is associated with the first term in the Volterra series that is a (linear) convolution integral operator. To analyze it, we refer again to our example of a circuit presented in Fig. 1. For this circuit, the first term of the Volterra series was formulated in (14). Assume now that the input signal $i_g(t)$ has, as assumed in the second Meyer's definition [14], the following form

$$K\sin\left(2\pi f_{s1}t+\varphi\right)=K\cos\left(2\pi f_{s1}t+\varphi-\pi/2\right)= \\ =\frac{K}{2}\Big(\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)+\exp\big(-j\left(2\pi f_{s1}t+\theta\right)\big)\Big)\ , \tag{23}$$

where the phase shift $\theta=\varphi-\pi/2$. By substituting (23) into (14), we get

$$\begin{aligned}\left(a_1\otimes i_g\right)(t)&=\int_{-\infty}^{\infty}a_1(\tau)\frac{K}{2}\Big(\exp\big(j\left(2\pi f_{s1}(t-\tau)+\theta\right)\big)+\\&+\exp\big(-j\left(2\pi f_{s1}(t-\tau)+\theta\right)\big)\Big)d\tau=\frac{K}{2}\Big[\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)\cdot\\&\cdot\int_{-\infty}^{\infty}a_1(\tau)\exp\left(-j2\pi f_{s1}\tau\right)d\tau+\exp\big(-j\left(2\pi f_{s1}t+\theta\right)\big)\cdot\\&\cdot\int_{-\infty}^{\infty}a_1(\tau)\exp\left(-j2\pi\left(-f_{s1}\right)\tau\right)d\tau\Big]=\frac{K}{2}\Big[\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)\cdot\\&\cdot A_1\left(f_{s1}\right)+\exp\big(-j\left(2\pi f_{s1}t+\theta\right)\big)\cdot A_1\left(-f_{s1}\right)\Big]\ .\end{aligned} \tag{24}$$

In (24), the definition of the one-dimensional Fourier transform has been used. Moreover, we have used a shorter notation for the argument in $A_1(\cdot)$. That is we have written $A_1\left(f_{s1}\right)$ instead of $A_1\left(j2\pi f_{s1}\right)$ and $A_1\left(-f_{s1}\right)$ instead of $A_1\left(-j2\pi f_{s1}\right)$, respectively. We will use this shorter notation consequently in what follows.

Further, note that after some algebraic manipulations (24) can be rewritten as

$$\begin{aligned}\left(a_1\otimes i_g\right)(t)&=\frac{K}{2}\left|A_1\left(f_{s1}\right)\right|\Big[\exp\Big(j\big(2\pi f_{s1}t+\theta+\varphi_{A_1(f_{s1})}\big)\Big)\cdot\\&\cdot A_1\left(f_{s1}\right)+\exp\Big(-j\big(2\pi f_{s1}t+\theta+\varphi_{A_1(f_{s1})}\big)\Big)\Big]=\\&=K\cdot\left|A_1\left(f_{s1}\right)\right|\cos\big(2\pi f_{s1}t+\theta+\varphi_{A_1(f_{s1})}\big)=\\&=\mathrm{Re}\Big\{A_1\left(f_{s1}\right)\cdot\Big(K\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)\Big)\Big\}\end{aligned} \tag{25}$$

because $\left[A_1\left(f_{s1}\right)\right]^*=A_1\left(-f_{s1}\right)$ holds. Moreover, $\left|A_1\left(f_{s1}\right)\right|$ and $\varphi_{A_1(f_{s1})}$ in (25) mean the magnitude and phase, respectively, of the circuit linear transfer function $A_1(\cdot)$ calculated at the frequency f_{s1}.

Observe now that the relationships in (25) allow to formulate a definition of operator *o* in the following way

$$\begin{aligned}A_1\left(f_1\right)o\big(K\cdot\cos\left(2\pi f_{s1}t+\theta\right)\big)&\overset{df}{=}\\ \overset{df}{=}\ \mathrm{Re}\Big\{A_1\left(f_{s1}\right)\cdot\Big(K\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)\Big)\Big\}&\ .\end{aligned} \tag{26}$$

So, the descriptive version of the definition given by (26) will be as follows: Take the complex-valued function $K\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)$ instead of $K\cdot\cos\left(2\pi f_{s1}t+\theta\right)$ and calculate the circuit transfer function $A_1\left(f_1\right)$ at the frequency $f_1=f_{s1}$ of the above cosine function. Multiply then $K\exp\big(j\left(2\pi f_{s1}t+\theta\right)\big)$ by $A_1\left(f_{s1}\right)$ and take finally the real part of this product. As a result we get $K\cdot\left|A_1\left(f_{s1}\right)\right|\cos\big(2\pi f_{s1}t+\theta+\varphi_{A_1(f_{s1})}\big)$ as required by the Meyer's second definition [14].

So, we can say that it is possible to formulate a mathematically precise definition of the operator *o* for the linear case, which fulfills the Meyer's postulate [14]. This definition is specified by (26).

In what follows, let us check whether we can get a similar result for the strictly nonlinear cases for which $n>1$. To this end, let us start with $n=2$. In this case, the second Meyer's definition [14] uses the sinusoid

$$K\sin\left(2\pi\left(f_{s1}+f_{s2}\right)t+\varphi\right)=K\cos\left(2\pi\left(f_{s1}+f_{s2}\right)t+ +\varphi-\pi/2\right)=\frac{K}{2}\left(\exp\left(j\left(2\pi\left(f_{s1}+f_{s2}\right)t+\theta\right)\right)+ +\exp\left(-j\left(2\pi\left(f_{s1}+f_{s2}\right)t+\theta\right)\right)\right) \quad (27)$$

of frequency $\left(f_{s1}+f_{s2}\right)$. To simplify derivation, let use for it a shorter notation $f_s=\left(f_{s1}+f_{s2}\right)$ in what follows. Further, according to the aforementioned definition, we apply the signal defined by (27) in the second component of a Volterra series [13] describing the nonlinear circuit of our example. This component has the following form

$$\left(a_2\otimes i_g^2\right)(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_2\left(\tau_1,\tau_2\right)i_g\left(t-\tau_1\right)i_g\left(t-\tau_2\right)d\tau_1 d\tau_2 \ . \quad (28)$$

In (28), $\left(a_2\otimes i_g^2\right)(t)$ means a two-dimensional convolution between a function a_2 of two time variables and a function i_g of one time variable.

By introducing i_g given by (27) into (28), we get

$$\left(a_2\otimes i_g^2\right)(t)=\frac{K^2}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_2\left(\tau_1,\tau_2\right)\left(\exp\left(j\left(2\pi f_s\left(t-\tau_1\right)+\theta\right)\right)+ +\exp\left(-j\left(2\pi f_s\left(t-\tau_1\right)+\theta\right)\right)\right)\left(\exp\left(j\left(2\pi f_s\left(t-\tau_2\right)+\theta\right)\right)+ +\exp\left(-j\left(2\pi f_s\left(t-\tau_2\right)+\theta\right)\right)\right)d\tau_1 d\tau_2=\frac{K^2}{4}\left[\exp\left(j\left(4\pi f_s t+2\theta\right)\right)\cdot \cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_2\left(\tau_1,\tau_2\right)\exp\left(-j2\pi f_s\tau_1\right)\exp\left(-j2\pi f_s\tau_2\right)d\tau_1 d\tau_2+ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_2\left(\tau_1,\tau_2\right)\exp\left(-j2\pi f_s\tau_1\right)\exp\left(-j2\pi\left(-f_s\right)\tau_2\right)d\tau_1 d\tau_2+ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_2\left(\tau_1,\tau_2\right)\exp\left(-j2\pi\left(-f_s\right)\tau_1\right)\exp\left(-j2\pi f_s\tau_2\right)d\tau_1 d\tau_2+ +\exp\left(-j\left(4\pi f_s t+2\theta\right)\right)\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a_2\left(\tau_1,\tau_2\right)\exp\left(-j2\pi\left(-f_s\right)\tau_1\right)\cdot \cdot\exp\left(-j2\pi\left(-f_s\right)\tau_2\right)d\tau_1 d\tau_2\right] \ . \quad (29)$$

Applying then a two-dimensional Fourier transformation in (29) leads to

$$\left(a_2\otimes i_g^2\right)(t)=\frac{K^2}{4}\left[\exp\left(j\left(4\pi f_s t+2\theta\right)\right)\cdot \cdot A_2\left(f_s,f_s\right)+A_2\left(f_s,-f_s\right)+A_2\left(-f_s,f_s\right)+ +\exp\left(-j\left(4\pi f_s t+2\theta\right)\right)\cdot A_2\left(-f_s,-f_s\right)\right] \ . \quad (30)$$

The relationship (29) can be made more compact by using the following equalities [13]: $A_2\left(-f_s,-f_s\right)=\left[A_2\left(f_s,f_s\right)\right]^*$ and $A_2\left(-f_s,f_s\right)=\left[A_2\left(f_s,-f_s\right)\right]^*$ in it. Applying this, we obtain

$$\left(a_2\otimes i_g^2\right)(t)=\frac{K^2}{2}\left[\left|A_2\left(f_s,f_s\right)\right|\cos\left(4\pi f_s t+2\theta+\varphi_{A_2\left(f_s,f_s\right)}\right)+ +\mathrm{Re}\left(A_2\left(f_s,-f_s\right)\right)\right]=\frac{K^2}{2}\left[\mathrm{Re}\left(A_2\left(f_s,f_s\right)\exp\left(4\pi f_s t+2\theta\right)\right)+ +\mathrm{Re}\left(A_2\left(f_s,-f_s\right)\right)\right]=\frac{K^2}{2}\mathrm{Re}\left[A_2\left(f_s,f_s\right)\exp\left(4\pi f_s t+2\theta\right)+ +A_2\left(f_s,-f_s\right)\right] \ . \quad (31)$$

Finally, let us introduce $f_s=\left(f_{s1}+f_{s2}\right)$ and $\theta=\varphi-\pi/2$ in the first equality in (31). It can be rewritten then as

$$\left(a_2\otimes i_g^2\right)(t)=\frac{K^2}{2}\left|A_2\left(f_{s1}+f_{s2},f_{s1}+f_{s2}\right)\right|\cdot \cdot\sin\left(2\pi\left(2f_{s1}+2f_{s2}\right)t+\left(2\varphi-\pi/2\right)+\varphi_{A_2\left(f_{s1}+f_{s2},f_{s1}+f_{s2}\right)}\right)+ +\frac{K^2}{2}\mathrm{Re}\left(A_2\left(f_{s1}+f_{s2},-\left(f_{s1}+f_{s2}\right)\right)\right) \ . \quad (32)$$

But, according to the Meyer's second definition [14], we should get in this case the following result

$$K\cdot\left|A_2\left(f_{s1},f_{s2}\right)\right|\cdot\sin\left(2\pi\left(f_{s1}+f_{s2}\right)t+\varphi+\varphi_{A_2\left(f_{s1},f_{s2}\right)}\right) \ . \quad (33)$$

Comparison of the expression on the right-hand side of equality (32) with the sinusoid given by (33) shows that they differ completely from each other. The amplitudes, frequencies, and phases of the sinusoids differ because we have $K^2/2\neq K$, $2\left(f_{s1}+f_{s2}\right)\neq\left(f_{s1}+f_{s2}\right)$, and $\left(2\varphi-\pi/2\right)+ +\varphi_{A_2\left(f_{s1}+f_{s2},f_{s1}+f_{s2}\right)}\neq\varphi+\varphi_{A_2\left(f_{s1},f_{s2}\right)}$, respectively. Moreover, the magnitude and phase of the circuit nonlinear transfer function $A_2(\cdot)$ occurring in (32) and (33) are calculated for different frequency pairs. That is for $\left(f_{s1}+f_{s2},f_{s1}+f_{s2}\right)$ and $\left(f_{s1},f_{s2}\right)$, accordingly. Furthermore, the expression in (32) contains a dc component, $\left(K^2/2\right)\cdot\mathrm{Re}\left(A_2\left(f_{s1}+f_{s2},-\left(f_{s1}+f_{s2}\right)\right)\right)$. Contrary to this, the dc component in (33) equals zero.

So, the results obtained above for the case $n=2$ show that it is not possible to construct an operator *o* in a similar way as it was done for the linear case (i.e. for $n=1$). In other words, as seen in (25), the last equality in it, which is the basis for the definition of an operator *o* for $n=1$, really holds. But, in opposite to this, the expression on the right-hand side of the equality (32), which could be also rewritten in the form of $\mathrm{Re}\{\cdot\}$, is not equal to the expression given by (33).

Analyzing the derivations that led to (32) and the form of the expression in (33), we come easily to the conclusion that the same holds also for the cases with $n>2$. That is it is not possible to construct an operator *o* for these cases in a similar way as it was done for the linear case.

Finally, we conclude that the second Meyer's definition of the operator *o* is faulty, too.

V. CONCLUSIONS

It has been shown that the definition of the operator *o* given in [3] was faulty. The needed corrections have been carried out and explained in section III of this paper. A new interpretation of the operator *o* (its second definition), communicated to the author of this paper in [14], has been discussed here, too. It has been shown that only a part of this definition that regards a linear circuit part has sense. Its remaining items are faulty.

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