

Spectrum Aliasing Does Occur Only in Case of Non-ideal Signal Sampling

Andrzej Borys

Abstract—In this paper, it has been shown that the spectrum aliasing and folding effects occur only in the case of non-ideal signal sampling. When the duration of the signal sampling is equal to zero, these effects do not occur at all. In other words, the absolutely necessary condition for their occurrence is just a nonzero value of this time. Periodicity of the sampling process plays a secondary role.

Keywords—Signal sampling, occurrence of spectrum aliasing and folding, modelling of non-ideal signal sampling operation

I. INTRODUCTION

THE author of this paper has shown in a previous one [1] that such phenomena as spectrum aliasing and folding (as illustrated in Fig. 1) do not occur in the case of an ideal signal sampling. Here, he continues this topic by considering what happens in a real, non-ideal case of performing the signal sampling operation.

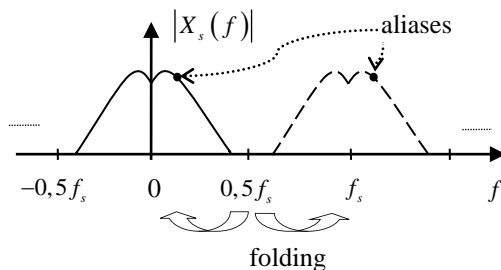


Fig. 1. Illustration to notions of aliasing, aliases, and folding in an example spectrum $X_s(f)$ of a sampled bandlimited signal. Figure taken from [1].

In Fig. 1, $X_s(f)$ means an example spectrum of a bandlimited signal and f_s the sampling frequency used.

It has been shown in [1] that the description of a sampled signal (which was obtained as a result of performing the sampling operation ideally) by a Dirac comb multiplied by its continuous time version leads to occurrence of artifacts in its spectrum. There is, however, a simple method to avoid this. It relies, as shown in [1], on describing a sampled signal in form of the so-called Kronecker comb [1] multiplying its analog version (that is the signal before sampling).

So, because of this reason, we extend here our considerations from [1] to the case of non-ideal sampling using

The author is with the Department of Marine Telecommunications, Faculty of Electrical Engineering, Gdynia Maritime University, Gdynia, Poland; (e-mail: a.borys@we.umg.edu.pl).

solely the latter model of sampling. The model of signal sampling that utilizes the Dirac comb is obviously not a natural one and can lead to errors, as shown in [1]. Therefore, one can expect that these errors, occurring in the ideal case of sampling, can also propagate into the non-ideal case (considered in this paper).

However, let us repeat first a basic material from [1] regarding the description of our ideal model of the signal sampling operation in terms of the Kronecker functions and the Kronecker comb [1]. This material will be needed for understanding all the derivations presented in the next sections, which will lead to obtaining an extended model. And, for this purpose, let us start with Fig. 2.

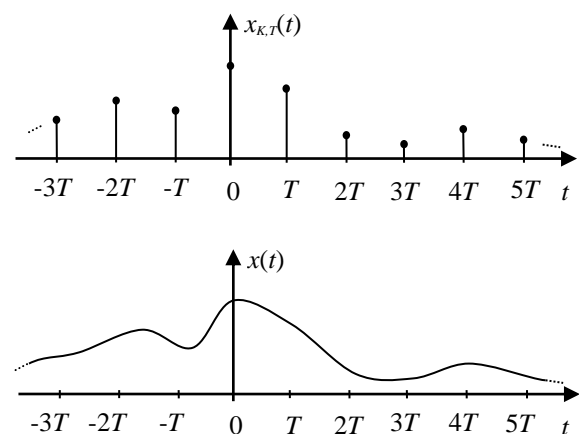


Fig. 2. Example sampled signal representation (upper curve) in form of a series of signal samples occurring uniformly on the continuous time axis in distance of T from each other, and its un-sampled version (lower curve), where t stands for a continuous time variable. Figure also taken from [1].

Note first that the notation used in Fig. 2 is the same as in [1]: $x(t)$ means a bandlimited signal in the continuous time domain and $x_{k,T}(t)$ its sampled version – also in the continuous time domain. T in Fig. 2 stands, obviously, for a sampling period but t is a continuous time variable. Moreover, $T = 1/f_s$ holds.

The sampled signal in Fig. 2 (upper curve) is modeled as a series of columns of different heights, which are proportional to the values of the signal samples at the corresponding time instants. And, these are finite numbers, what is obviously opposite of modeling the sampled signal as a series of the Dirac impulses. Although, the latter ones are then multiplied by finite numbers (being the values of the signal samples), but they still remain Dirac impulses.

It is also worth noting that the modeling of a sampled signal, in the continuous time domain, with the use of Dirac impulses is a generally agreed canon in the literature; see, for example, [2]–[4]. However, as shown recently in [1], the multiplication of the signal samples (columns) as in Fig. 2 (upper curve) by the Dirac impulses is not only superfluous but it also leads to a defective representation of the sampled signal spectrum. Therefore, it seems that some revisions of the theory of signal sampling will follow.

The description of the ideal means of signal sampling illustrated graphically in Fig. 2 can be also expressed analytically. To this end, let us recall the definition of a time-shifted Kronecker time function $\delta_{i,t/T}(t)$ that was defined in [1] as

$$\delta_{i,r}(t) = \delta_{i,t/T}(t) = \begin{cases} 1 & \text{if } i = r = t/T \\ 0 & \text{otherwise} \end{cases}, \quad (1)$$

with $r = t/T$ being a real number (that is r belongs to the set \mathbb{R} denoting the set of real numbers). So, with this, $\delta_{i,r}$ in (1) stands for a slightly modified standard Kronecker delta symbol in which now the second index r is a real-valued one. And only when it becomes an integer equal to i , the function $\delta_{i,r}(t) = \delta_{i,t/T}(t)$ differs from zero (assumes the value 1).

It is clear that $\delta_{i,t/T}(t)$ given by (1) is a function of a continuous time variable t . It has been named a time-shifted Kronecker time function in [1] and derives from another Kronecker time function called a basic Kronecker time function $\delta_{0,t/T}(t)$ in [1]. The time shift between the above functions equals iT seconds; for more material regarding the Kronecker time functions see, however, [1].

The function $\delta_{i,t/T}(t)$ for $i = 1$ is illustrated in Fig. 3.

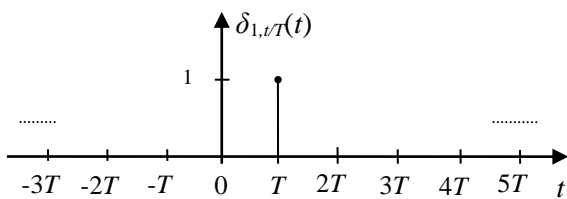


Fig. 3. Illustration of the function $\delta_{i,t/T}(t)$ for the parameter $i = 1$. Figure also taken from [1].

Let us now repeat the defining equation of the so-called Kronecker comb, $\delta_{K,T}(t)$, (so named in [1]; in our model, it is a counterpart of the Dirac comb [2]–[4]). This equation has the following form:

$$\delta_{K,T}(t) = \sum_{k=-\infty}^{\infty} \delta_{k,t/T}(t), \quad (2)$$

where the first index K at $\delta_{K,T}(t)$ stands for the name of Kronecker, but the second one, T , means a repetition period.

The Kronecker comb given by (2) is illustrated in Fig. 4.

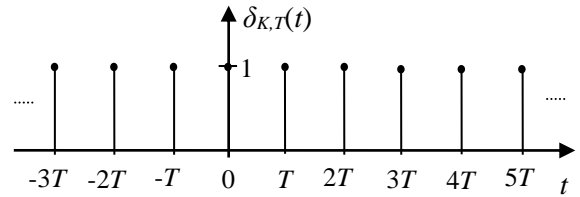


Fig. 4. Visualization of the Kronecker comb given analytically by (2). Figure taken also from [1].

In the next step, observe that using (1) and (2) we are now able to describe the sampled signal, $x_{K,T}(t)$, which is depicted in Fig. 2 (upper curve), analytically in the following way:

$$x_{K,T}(t) = \sum_{k=-\infty}^{\infty} x(kT) \delta_{k,t/T}(t) = \delta_{K,T}(t) \cdot x(t), \quad (3)$$

where, similarly as before, the first index K at $x_{K,T}(t)$ stands for the name of Kronecker, but the second one, T , means the sampling period. For more details regarding the description (3), see [1].

This model of the signal sampling performed ideally, illustrated graphically with the help of Figs. 2, 3, and 4 as well as expressed analytically in a compact form by (3), will be the subject of its extension to the non-ideal case in the next sections.

The remainder of this paper is organized as follows. In the next section, we develop graphical and analytical descriptions of a sampled signal in the case of non-ideal sampling. Whereby the latter is modeled here by assuming that the sampling lasts a finite time τ . In effect, we get signal impulses, which can be viewed as “smeared” signal samples, in place of numbers assigned to the corresponding points on the continuous time axis (that are finally replaced by the Dirac deltas in the ideal model of signal sampling used in the literature [2]–[4]). Next, the spectrum of the smeared sampled signal is calculated. In Section IV, this spectrum is discussed in detail for two limiting values of the model parameter τ . As a result, a few very valuable remarks and conclusions are formulated. The paper ends with a final conclusion.

II. GRAPHICAL AND ANALYTICAL DESCRIPTION OF A SAMPLED SIGNAL IN THE CASE OF NON-IDEAL SAMPLING

In this section, we model the non-ideality of the signal sampling by simply assuming that this operation lasts a finite time, say, τ seconds. In other words, we assume here that the signal sampling is not carried out immediately but it needs some time to be performed. And, the simplest way to take this fact into account in our model seems to be the one that is sketched graphically in Fig. 5.

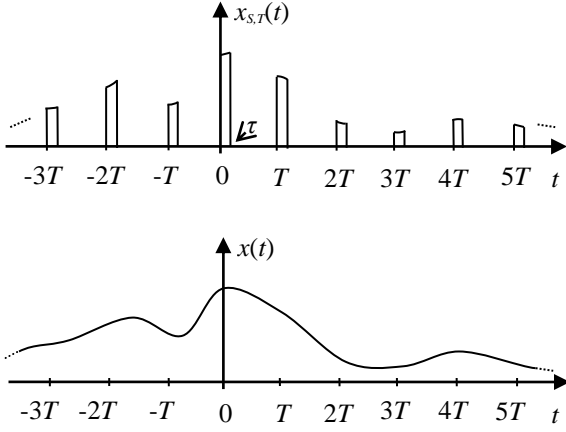


Fig. 5. Representation of an example, not ideally sampled signal (upper curve) in form of a series of not ideal samples of the signal shown below it (lower curve). The lower curve is the same as in Fig. 2.

Note that the upper curve of Fig. 5, $x_{s,T}(t)$, can be considered as a “smeared” kind of the corresponding one in Fig. 2, $x_{k,T}(t)$, in which now each “ideal sample value” is “smeared” on a time segment of the length of τ seconds. Whereby, here, modeling of the “smearing” operation relies simply on cutting a segment of the signal lasting from a time instant kT , $k = \dots -1, 0, 1, \dots$, to the instant $kT + \tau$, and next taking it instead of the ideal sample value $x(kT)$.

To proceed further, let us now extend the definition of the time-shifted Kronecker time function $\delta_{i,t/T}(t)$ given by (1) to the “smeared” case (in the sense as described above). So, to this end, let us define a real-valued index that includes a condition. We do this by defining the following function, $\alpha(t, \tau)$:

$$\alpha(t, \tau) = \begin{cases} i & \text{if } iT \leq t \leq iT + \tau \\ t/T & \text{otherwise} \end{cases} \quad (4)$$

And, in the next step, see that this function can play a role of the real-valued index r in $\delta_{i,r}(t)$. Hence, if we introduce $\alpha(t, \tau)$ given by (4) into $\delta_{i,r}(t)$ defined by (1), we get an appropriate time-shifted Kronecker time function for the description of our “smeared case” considered now. It will have then the following form:

$$\delta_{i,\alpha(t,\tau)}(t) = \begin{cases} 1 & \text{if } iT \leq t \leq iT + \tau \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

The function $\delta_{i,\alpha(t,\tau)}(t)$ for $i = 1$ is illustrated in Fig. 6.

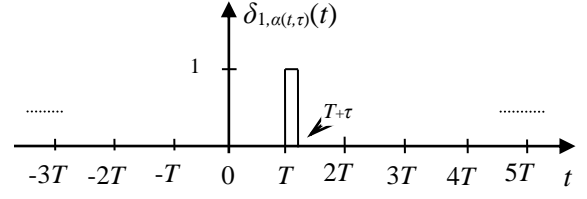


Fig. 6. Illustration of the function $\delta_{i,\alpha(t,\tau)}(t)$ for the parameter $i = 1$.

Note now that having defined a “smeared” version of the time-shifted Kronecker time function $\delta_{i,t/T}(t)$, that is $\delta_{i,\alpha(t,\tau)}(t)$ given by (5), we are able to formulate a “smeared” Kronecker comb. Further, let us denote it using the following symbol: $\delta_{s,T}(t)$, where the first index, S , stands now for the word “smeared”, but the second one, T , means a repetition period (as before). In analogy to (2), it can be expressed as

$$\delta_{s,T}(t) = \sum_{k=-\infty}^{\infty} \delta_{k,\alpha(t,\tau)}(t) \quad (6)$$

In the next step, similarly as in the previous case, observe that using (5) and (6) we are able to describe analytically the sampled “smeared” signal, $x_{s,T}(t)$, depicted in Fig. 5 (upper curve), in the following way:

$$\begin{aligned} x_{s,T}(t) &= x(t) \cdot \delta_{s,T}(t) = \sum_{k=-\infty}^{\infty} x(t) \delta_{k,\alpha(t,\tau)}(t) = \\ &= \delta_{s,T}(t) \cdot x(t) \end{aligned} \quad (7)$$

where, as before, the first index, S , at $x_{s,T}(t)$ stands for the word “smeared”, but the second one, T , means the sampling period.

III. SPECTRUM OF THE SMEARED SAMPLED SIGNAL IN THE MODEL PROPOSED

In the previous section, we developed a function, denoted as $x_{s,T}(t)$, in the continuous time domain that describes the signal sampled non-ideally (modeled here as a smeared version of the one sampled ideally). It is expressed in (7) as a multiplication of two “well-defined” functions. (Here, under the term “well-defined” functions, we understand that they are not Dirac distributions or sums of them as well as they do not represent single finite values or sums of them separated on the time axis). So, we are now in a position to calculate its spectrum.

In our calculations presented here, we assume that the signal $x(t)$ is an energy, bandlimited one. Furthermore, see the signal (function) $\delta_{s,T}(t)$ is a periodic one. Therefore, it can be expanded in a Fourier series; that is it can be expressed in the following form:

$$\delta_{s,T}(t) = \sum_{k=-\infty}^{\infty} a_k \exp(j2\pi k t/T) \quad (8)$$

where j means $j = \sqrt{-1}$ and the coefficients a_k (Fourier series coefficients) are given by

$$\begin{aligned} a_k &= \frac{1}{T} \int_0^T \delta_{s,T}(t) \exp(-j2\pi k t/T) dt = \\ &= \frac{1}{T} \int_0^\tau 1 \cdot \exp(-j2\pi k t/T) dt = \\ &= \frac{1}{T(-j2\pi k/T)} \exp(-j2\pi k t/T) \Big|_0^\tau \end{aligned} \quad (9)$$

So, we get finally from (9) the following:

$$a_k = \frac{1}{j2\pi k} [1 - \exp(-j2\pi k \tau/T)] \quad (10)$$

Note now that we can perform some algebraic manipulations in (10) as follows below to get

$$\begin{aligned} a_k &= \frac{1}{j2\pi k} \exp(-j2\pi k \tau/(2T)) \cdot \\ &\cdot [\exp(j2\pi k \tau/(2T)) - \exp(-j2\pi k \tau/(2T))] = \\ &= \frac{1}{j2\pi k} \exp(-j\pi k \tau/T) \cdot 2 \sin(\pi k \tau/T) = \\ &= \frac{\tau}{T} \exp(-j\pi k \tau/T) \cdot \text{sinc}(\pi k \tau/T), \end{aligned} \quad (11)$$

where the function $\text{sinc}(x)$ is defined as

$$\text{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } |x| \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \quad (12)$$

In the next step, let us introduce $\delta_{s,T}(t)$ given by (8) into (7) – having in mind that the Fourier coefficients in (8) possess the form achieved in the last line of (11). So, we have now

$$x_{s,T}(t) = \sum_{k=-\infty}^{\infty} a_k \exp(j2\pi k t/T) x(t) \quad (13)$$

And, we calculate, in what follows, the spectrum of the signal $x_{s,T}(t)$ as expressed in (13); let us denote it by $X_{s,T}(f)$. Then, we get

$$\begin{aligned} X_{s,T}(f) &= \int_{-\infty}^{\infty} x_{s,T}(t) \exp(-j2\pi f t) dt = \\ &= \int_{-\infty}^{\infty} x(t) \sum_{k=-\infty}^{\infty} a_k \exp(-j2\pi(f - k/T)t) dt \end{aligned} \quad (14)$$

where f means a continuous frequency variable.

After changing the order of integration and summation in (14), we obtain

$$\begin{aligned} X_{s,T}(f) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} x(t) \exp(-j2\pi(f - k/T)t) dt = \\ &= \sum_{k=-\infty}^{\infty} a_k X(f - k/T), \end{aligned} \quad (15)$$

where $X(\cdot)$ stands for the Fourier transform (spectrum) of the signal $x(t)$.

The result given by (15) is a key outcome in this paper. It simply shows that there occur aliasing and folding effects (as illustrated in Fig. 1) in the case of non-ideal signal sampling (modeled by means of a smearing of signal samples – as illustrated in Fig. 5, on the upper curve there). That is this is unlike in the case of an ideal signal sampling, in which, as shown in [1] and [5], no aliasing and folding of the spectrum occurs.

Further, the spectrum aliasing and folding effects occur only when the parameter τ in our model of the non-ideal sampling is larger than zero seconds. Or, in other words, these effects are present only when “smearing” of signal samples lasts some time; that is the signal sampling does not happen immediately.

In view of these observations, the “smearing” of signal samples, which obviously occurs in practical sampling, must be considered as a necessary condition of occurrence of the spectrum aliasing and folding effects.

Finally at the end of this section, we would like to make yet one comment regarding our derivation of the result in (15). Namely in our derivations from (8) to (15), we have tacitly assumed that the Fourier series given by (8) is a convergent one for all the points on the time axis. So, here, for having such a series, we need to modify slightly the function given by (5) in its discontinuity points. And, we do this by re-defining it as follows:

$$\delta_{i,\alpha(t,\tau)}^{\text{modif}}(t) = \text{modif}(\delta_{i,\alpha(t,\tau)}(t)) = \begin{cases} 1 & \text{if } iT < t < iT + \tau \\ 1/2 & \text{if } t = iT \\ 1/2 & \text{if } t = iT + \tau \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

Note that (16) clearly shows what the modified function $\delta_{i,\alpha(t,\tau)}^{\text{modif}}(t)$ does mean, and what the $\text{modif}(\delta_{i,\alpha(t,\tau)}(t))$ operation on the function $\delta_{i,\alpha(t,\tau)}(t)$ does do. Further, the periodic function $\delta_{i,\alpha(t,\tau)}^{\text{modif}}(t)$ satisfies the Dirichlet condition [4] regarding its values at the discontinuity points to expand it in a Fourier series convergent at every point.

So, in summary, we can say that with $\delta_{i,\alpha(t,\tau)}^{\text{modif}}(t)$ instead of $\delta_{i,\alpha(t,\tau)}(t)$ in (6) our derivations from (8) to (15) are then – mathematically – precisely correct. Furthermore, we can also say that the above slight correction in our model does not change in fact nothing relevant in it.

IV. CONSIDERATION OF TWO LIMITING VALUES OF THE PARAMETER TAU

Let us check in this section two interesting cases in our way of modeling the non-ideal signal sampling. These are the following ones:

1. the parameter τ (tau) defined in a description of the upper curve of Fig. 5 (see the text below this figure) approaches the zero value or it equals zero;
2. the parameter τ (tau) approaches the value of T denoting the sampling period or it equals the value of T .

Observe that the values 0 and T are the extremal ones, which the parameter τ can assume. The first one is the smallest, but the second the largest possible.

Consider first the first case mentioned above. And, let us start by noting that analyzing it will provide us with an answer to the question of the possibility of obtaining the following result:

$$X_s(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - k/T) \quad (17)$$

from (15), where $X_s(f)$ in (17) means the spectrum of the signal sampled ideally and modeled with the use of a Dirac comb (denote it here as $x_s(t)$). In other words, we ask here whether it is possible to achieve this highly celebrated and commonly used [2]–[4] expression (17) for describing the spectrum aliasing and folding effects in the case of an ideal signal sampling – from our model – by consideration of the limiting value of $\tau = 0$ in it. Or, otherwise, shortly: does any of the following relations:

$$X_{s,T}(f) \xrightarrow{\text{for } \tau \rightarrow 0} X_s(f) \quad (18)$$

given by (15) given by (17)

or

$$X_{s,T}(f) \stackrel{\text{for } \tau=0}{=} X_s(f) \quad (19)$$

given by (15) given by (17)

hold (is true)?

Let us now check the validity of (19). To this end, consider the coefficients a_k given by (11); and, see that it follows from (11) that

$$\begin{aligned} a_k &= \frac{0}{T} \exp(-j\pi k 0/T) \cdot \text{sinc}(\pi k 0/T) = \\ &= 0 \cdot 1 \cdot 1 = 0 \quad \text{for all indices } k. \end{aligned} \quad (20)$$

So, according to (15), $X_{s,T}(f; \tau = 0) \equiv 0$, which also does mean that (19) does not hold.

Note that we get a similar result in the procedure of checking of (18) because the following:

$$\begin{aligned} |a_k| &= \frac{\tau}{T} \left| \exp(-j\pi k \tau/T) \cdot \text{sinc}(\pi k \tau/T) \right| \leq \frac{\tau}{T} \xrightarrow{\tau \rightarrow 0} \\ &\rightarrow 0 \quad \text{for all indices } k. \end{aligned} \quad (21)$$

holds. That is we have $X_{s,T}(f; \tau \rightarrow 0) \rightarrow 0$ in this case. And, obviously, this prevents (18) to be true.

So, in summary, we conclude that derivation of the spectrum formula [2]–[4] for the signal sampled ideally and modeled with the help of the Dirac comb formalism – from the model taking into account the sampling operation non-ideality – is not possible. Even worse, the sampled signal spectrum $X_{s,T}(f; \tau = 0) \equiv 0$ or $X_{s,T}(f; \tau \rightarrow 0) \rightarrow 0$. That is it disappears.

Note, in the above context, that just to avoid the latter phenomenon a new reasonable model of an ideal signal sampling operation has been invented in [1].

Also, it is worth noting that it is possible to rescue the procedure of getting the formula (17) from the formula (15), which was discussed above, by performing a normalization of the expression (15) and putting then $\tau = 0$ or going with τ to zero in the normalized expression.

To explain what we mean under the above, let us consider first the normalization of (15). We normalize (15) here with respect to the parameter τ , what means dividing (15) by τ . As a result, we get then

$$X_{sn,T}(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} b_k X(f - k/T) \quad (22)$$

with the new coefficients b_k dependent upon the parameter τ as shown below

$$b_k = \exp(-j\pi k \tau/T) \cdot \text{sinc}(\pi k \tau/T). \quad (23)$$

$X_{sn,T}(f)$ in (22) stands for $X_{s,T}(f)$ that is normalized with respect to the parameter τ .

At this point, observe now that (22) resembles (17) very well, however, the coefficients b_k in (22) are still not ones – as in (17). But, note that arriving at this can be easily achieved by substitution of $\tau = 0$ in (23). Then, all the coefficients b_k are equal to 1.

It is interesting to note that the case of going with τ to zero in the normalized expression is not identical with the substitution of $\tau = 0$ in (23). To see this, consider the absolute value of b_k given by (23), that is

$$|b_k| = \left| \exp(-j\pi k \tau/T) \cdot \text{sinc}(\pi k \tau/T) \right| = \left| \text{sinc}(\pi k \tau/T) \right|. \quad (24)$$

It follows from (24) that for any arbitrarily small τ one can find such a $k_{0\tau} > 0$ that for all $k \geq k_{0\tau}$ $|b_k|$ will be

significantly smaller than 1. (Note that because of the symmetry of the function $\text{sinc}(x)$ around the point $x=0$ the same regards also the negative k 's in (24). That is those negative k 's for which $|k| \geq k_{0\tau}$ holds. This fact will be, obviously, taken into account in a final result, however, at the moment, we will operate with the positive k 's only, for the simplicity of notation.)

Therefore, getting (17) from (22) is generally not possible. However, it is always possible to find such a set of values of $k \in \{1, \dots, k_{1\tau}\}$, that is integers changing from 1 to, say, $k_{1\tau}$ dependent upon the value of the parameter τ , for which we will have approximately

$$|b_k| = |\text{sinc}(\pi k \tau / T)| \cong 1 \quad (25)$$

with an accuracy, say, η . That is, shortly, we will have a set of coefficients b_k satisfying the following inequality:

$$|1 - b_k| = |1 - \text{sinc}(\pi k \tau / T)| \leq \eta. \quad (26)$$

with $\pi k \tau / T < \pi$.

Next, see that the following:

$$\pi k \tau / T \leq \text{sinc}^{-1}(1 - \eta) = \theta < \pi. \quad (27)$$

can be easily derived from (26), where the parameter θ is defined as indicated in (27). And, let us now illustrate this final result. For example, assuming the accuracy level η equal to about 1/10, what gives θ equal to about $\pi/4$, and assuming that the ratio of the ‘‘sample smearing’’ to the sampling period, $\tau/T = 0,01$, we get $k \leq 25$ from (27). So, this allows us to assume here the parameter $k_{1\tau}$ (defined above) to be equal to 25. Furthermore, if we restrict ourselves in this example to consideration of the first $k_{1\tau}$ positive indices plus zero index and plus $k_{1\tau}$ negative indices in (22), then its form will almost perfectly resemble that of (17). In other words, the following:

$$\begin{aligned} X_{S_n, T}(f; k_{1\tau}) &= \frac{1}{T} \sum_{k=-k_{1\tau}=-25}^{k_{1\tau}=25} b_k X(f - k/T) \cong \\ &\cong \frac{1}{T} \sum_{k=-k_{1\tau}=-25}^{k_{1\tau}=25} X(f - k/T) = X_s(f; k_{1\tau}) \end{aligned} \quad (28)$$

then holds approximately, where $X_{S_n, T}(f; k_{1\tau})$ and $X_s(f; k_{1\tau})$ are defined as indicated in (28). However, it does not mean at all that the validity of the approximate equality (28) guarantees, at the same time, that $X_{S_n, T}(f)$ given by (22) equals approximately $X_s(f)$ given by (17).

For better understanding of the above approach, let us try now to explain it in terms of the windowing operation that is

so very popular and widely used in the digital signal processing. In windowing, we ‘‘close’’ a part of a signal to process it according to some algorithm that is deduced from some theory. And, in the window, we have approximately what we want to have or what should be according to some theory. But, we do not care at all about what happens outside our window we work in. In many cases, however, this, what we have in the window, differs completely from what we have outside it. Further, we can shape the window to get a better approximation in it and enlarge its length to get a ‘‘larger amount’’ of a signal processed according to our wishes. But, always, not taking care of the signal outside the window.

It seems to us that the above description of the windowing operation describes quite well the procedure with $\tau \rightarrow 0$, discussed before.

Now, in summary, we can conclude that applying some tricks – relying on normalization of an expression going to zero for $\tau \rightarrow 0$, just to avoid this vanishing, and afterwards putting $\tau = 0$ (or calculating a limit for $\tau \rightarrow 0$) in the resulting expression – leads to obtaining a result resembling (17) (in an exact or in an approximate form). So, only the application of these tricks mentioned, which are hard to justify in a reasonable way, enabled getting an awaited result. And, note that the same situation occurs in the case of modeling a sampled signal by a series of Dirac impulses. Modeling signal samples by Dirac impulses of zero duration has no physical justification. Physically, it is not possible to generate impulses of zero duration. Therefore, the sampled signal modeling with the use of Dirac deltas should be rather treated as an inappropriate way (not a correct one).

Consider now the second extremal case mentioned at the beginning of this section, namely the case in which the parameter τ approaches the value of T . To this end, substitute simply $\tau = T$ into the formula (11) expressing the coefficients a_k . We get then the following:

$$\begin{aligned} a_k &= \exp(-j\pi k) \cdot \text{sinc}(\pi k) = \text{sinc}(\pi k) = \\ &= \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}. \end{aligned} \quad (29)$$

In the next step, applying (29) in (15) gives

$$X_{S, T}(f) = X(f - 0/T) = X(f), \quad (30)$$

and obviously this result is as expected in this case.

Finally, at the end of this section, consider also a little bit more complicated variant of the latter one. That is, assume now that $\tau \rightarrow T$ in (15) with the coefficients a_k given by (11). And, we look for a condition, which will allow us to write approximately the following:

$$X_{S, T}(f) \cong a_0 X(f - 0/T) = X(f) \quad (31)$$

in the sense that all the absolute values of the coefficients

$a_k, k \neq 0$, can be regarded as being sufficiently lower than the value of $|a_0| = a_0 = 1$. That is all the spectrum components at larger frequencies than $|f| > 1/(2T)$ can be neglected with respect to $|X(f)|$. In other words, we start with

$$|a_0| - |a_{k \neq 0}| = 1 - |a_{k \neq 0}| > \gamma, \quad (32)$$

where γ means a minimal required difference between the absolute values of the coefficient a_0 and any other one, denoted here as $a_{k \neq 0}$.

Using (11) in (32) and performing then some algebraic manipulations, we get

$$(1-\gamma) > \frac{\tau}{T} \left| \text{sinc}(\pi k \tau / T) \right|. \quad (33)$$

In the next step, substituting into (33) $\tau = T - \varepsilon$, where ε means any small positive real number, we arrive at

$$(1-\gamma) > \frac{T-\varepsilon}{T} \left| \text{sinc}(\pi k (T-\varepsilon) / T) \right|. \quad (34)$$

So, after rearranging terms in (34), we obtain

$$(1-\gamma) > \left| \text{sinc}(\pi k - \pi k \varepsilon / T) \right| - \frac{\varepsilon}{T} \left| \text{sinc}(\pi k - \pi k \varepsilon / T) \right|. \quad (35)$$

Consider now the inequality

$$(1-\gamma) > \left| \text{sinc}(\pi k - \pi k \varepsilon / T) \right| \quad (36)$$

and observe that if (36) is satisfied, then (35) is satisfied, too. And, in what follows, we will consider (36); we will show that this inequality is satisfied when ε is chosen to be sufficiently small. In other words, we will show that independently of how small the value of $(1-\gamma)$ is, for example equal to 0,01, to 0,001 or even smaller, it will be always possible to find such a ε for which (36) will be satisfied (and, obviously, also for all smaller values of ε).

We will carry out this task in two steps. First, we observe that the function $\left| \text{sinc}(\pi k - \pi k \varepsilon / T) \right|$ is an oscillating function

going to zero when its argument $|\pi k - \pi k \varepsilon / T|$ is going to infinity. Therefore, we can say that for a given value of ε , say ε_0 , $\left| \text{sinc}(\pi k - \pi k \varepsilon_0 / T) \right| < (1-\gamma)$ holds, when the $|k|$ is chosen to be sufficiently large. So, denote by $|k_{\varepsilon_0 \gamma}|$ such a $|k|$ that for all $|k| \geq |k_{\varepsilon_0 \gamma}|$, for given γ and ε_0 , (36) is satisfied.

Next, after performing the above, consider now all the $|k|$'s for which we have $0 < |k| < |k_{\varepsilon_0 \gamma}|$. Probably, we will need, by a given value of γ , to make smaller the value of ε , which was chosen previously – to be able to satisfy (36). (Note that this will be always possible.) And, we do this.

Further, denote this value of ε as ε_m that ensures the above; and, obviously, the following relation: $\varepsilon_0 \geq \varepsilon_m$ will hold. Therefore, if we substitute ε_m into (36) valid for $|k| \geq |k_{\varepsilon_0 \gamma}|$ instead of ε_0 , this inequality will be all the more satisfied.

And, concluding all the latter derivations, we see that really (31) can be satisfied with any accuracy chosen when we let the parameter ε go to zero (that is if we let $\tau \rightarrow T$).

V. FINAL CONCLUSION

Once again, the main conclusion following from the results presented in this paper is that the absolutely necessary condition for the occurrence of the aliasing and folding effects in the sampled signal spectrum is a nonzero value of duration of the sampling operation. Obviously, any practical A/D sampler ensures this. Furthermore, in this context, the periodicity of the sampling process plays only a secondary role.

REFERENCES

- [1] A. Borys, "Spectrum aliasing does not occur in case of ideal signal sampling," *Intl Journal of Electronics and Telecommunications*, submitted for publication in October 2020.
- [2] A. V. Oppenheim, R. W. Schaffer, and J. R. Buck, *Discrete-Time Signal Processing*. New Jersey: Prentice Hall, 1998.
- [3] R. J. Marks II, *Introduction to Shannon Sampling and Interpolation Theory*, New York: Springer-Verlag, 1991.
- [4] R. N. Bracewell, *The Fourier Transform and Its Applications*, New York: McGraw-Hill, 2000.
- [5] A. Borys, "Filtering property of signal sampling in general and under-sampling as a specific operation of filtering connected with signal shaping at the same time," *Intl Journal of Electronics and Telecommunications*, vol. 66, no. 3, pp. 589-594, 2020.